2016 MAT – Q2 (3 pages; 3/11/23)

Solution

(i) $A\big(B(x)\big) = 2(3x + 2) + 1 = 6x + 5$

and $B\big(A(x)\big) = 3(2x + 1) + 2 = 6x + 5$

(ii) $A^2(x) = 2(2x + 1) + 1 = 2^2x + (2 + 1)$

$A^3(x) = 2[2^2x + (2 + 1)] + 1 = 2^3x + (2^2 + 2 + 1)$

And so $A^n(x) = 2^nx + (2^{n-1} + 2^{n-2} + \cdots + 2 + 1)$

$= 2^nx + \frac{2^n-1}{2-1} = 2^n(x + 1) - 1$

[This agrees with $A^2(x)$ and $A^3(x)$.]

(iii) First of all, by (i), a sequence such as $A(B\left(A\big(A(B(x))\big)\right))$, for example, is equal to

$$A\left(A\left(B\left(A\big(B(x)\big)\right)\right)\right)$$

$$= A\left(A\left(A\left(B\big(B(x)\big)\right)\right)\right)$$

$$= A^3(B^2(x))$$

And so any combination of p As and Q Bs will be equal to

$A^p(B^q(x))$

Now $B^2(x) = 3(3x + 2) + 2 = 3^2x + 3(2) + 2$

$B^3(x) = 3[3^2x + 3(2) + 2] + 2 = 3^3x + (3^2 + 3 + 1)(2)$

And so $B^n(x) = 3^nx + (3^{n-1} + 3^{n-2} + \cdots + 3 + 1)(2)$

$= 3^nx + \frac{(3^n-1)(2)}{3-1} = 3^n(x+1) - 1$

Then $A^p\left(B^q(x)\right) = 2^p(B^q(x) + 1) - 1$

$= 2^p([3^q(x+1) - 1] + 1) - 1$

$= 2^p3^q(x+1) - 1$

Given that $A^p\left(B^q(x)\right) = 108x + c$ (for all x),

it follows that $2^p3^q = 108$ and $2^p3^q - 1 = c$, so that $c = 107$

As $108 = 2^23^3$, the number of orders is the number of ways of choosing the 2 places for A out of the total of 5 places for A and B;

ie $\binom{5}{2} = \frac{5(4)}{2!} = 10$

[Note: This part of the question could have been answered just by noting that the coefficient of x had to be 2^p3^q. The derivation of

$A^p\left(B^q(x)\right) = 2^p3^q(x+1) - 1$ is only needed to establish c.]

(iv) As above, $c = 107$

(v) We require $[2^{m_1}3^{n_1}(x+1) - 1] + [2^{m_2}3^{n_2}(x+1) - 1]$

$+ \cdots + [2^{m_k}3^{n_k}(x+1) - 1] = 214x + 92$ (for all x)

Then $2^{m_1}3^{n_1} + 2^{m_2}3^{n_2} + \cdots + 2^{m_k}3^{n_k} = 214$ (*)

and $2^{m_1}3^{n_1} + 2^{m_2}3^{n_2} + \cdots + 2^{m_k}3^{n_k} - k = 92$,

which gives $k = 214 - 92 = 122$

As the m_i & n_i must be positive integers,

$2^{m_1}3^{n_1} + 2^{m_2}3^{n_2} + \cdots + 2^{m_k}3^{n_k} \geq 2(3)(122) = 732 > 214$

Thus (*) cannot be satisfied; ie no such m_i & n_i exist.

[The Official Sol'n says that the x coefficient of $A^{m_i}B^{n_i}$ can never be less than 2, but this seems to (incorrectly) allow $n_i = 0$ (with $m_i \geq 1$).]