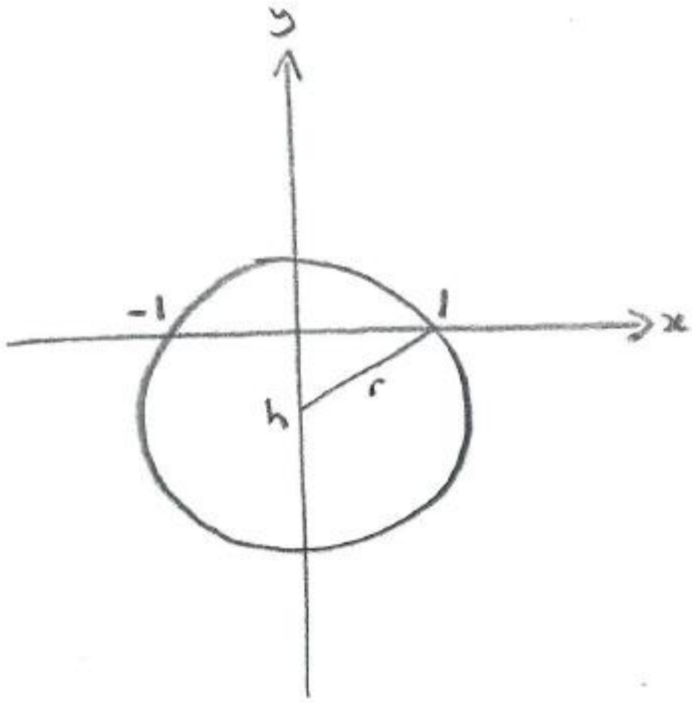
2015 MAT Paper - Q4 (5 pages; 24/9/20)

(i) [A diagram may reveal something that hasn't been considered.]



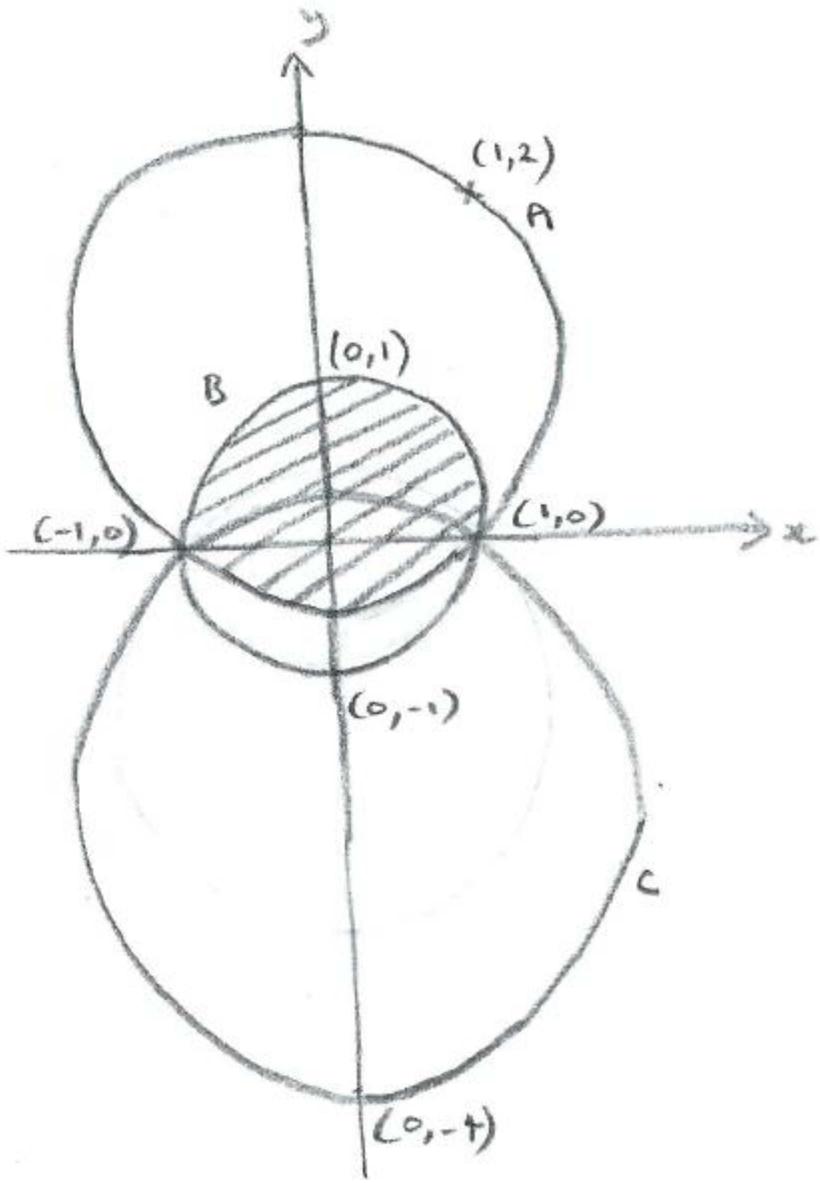
The diagram shows one possible configuration. We note that the centre of the circle will lie on the perpendicular bisector of the two points; ie the y-axis, and so $m = 0$.

The centre could also lie above the Origin, but in either case

$r = \sqrt{h^2 + 1}$

(ii) The centre of the circle lies on the y-axis and also on the perpendicular bisector of (for example) (1,0) & (x_0, y_0). Either $y_0 > 0$ or $y_0 < 0$, and from the diagram we can see that in each case the centre is uniquely defined as the intersection of these lines. The radius is then found from the centre and one of the 3 points, and so the circle is uniquely determined.

(iii) Referring to the diagram below, we only need to find the shaded region of B, A_1 say, as well as the area of A, A_2 say.



Then the lopsidedness of circle A is $1 - \frac{A_1}{A_2}$

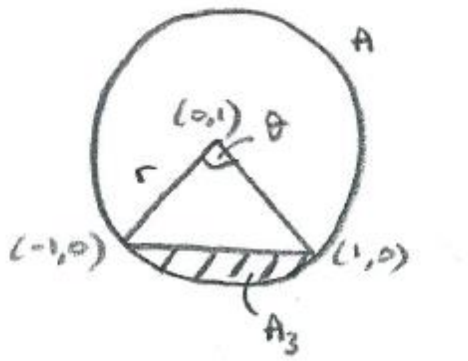
We note that the centre of circle A lies on the perpendicular bisector of the line joining (1,0) & (1,2), as well as being on the

y-axis; so the centre is $(0,1)$, and the radius is seen to be $\sqrt{2}$

Hence $A_2 = 2\pi$

To find A_1, we need to determine the area of the segment of A, A_3 say, that lies below the y-axis. Referring to the diagram below,

$A_3 = \frac{1}{2}r^2(\theta - sin\theta)$, where $r = \sqrt{2}$ and θ is seen to be $\frac{\pi}{2}$ (considering one of the right-angled triangles formed by bisecting the angle θ; the sides of which are 1, 1 & $\sqrt{2}$)



So $A_3 = \frac{1}{2}(2)\left(\frac{\pi}{2} - 1\right) = \frac{\pi}{2} - 1$

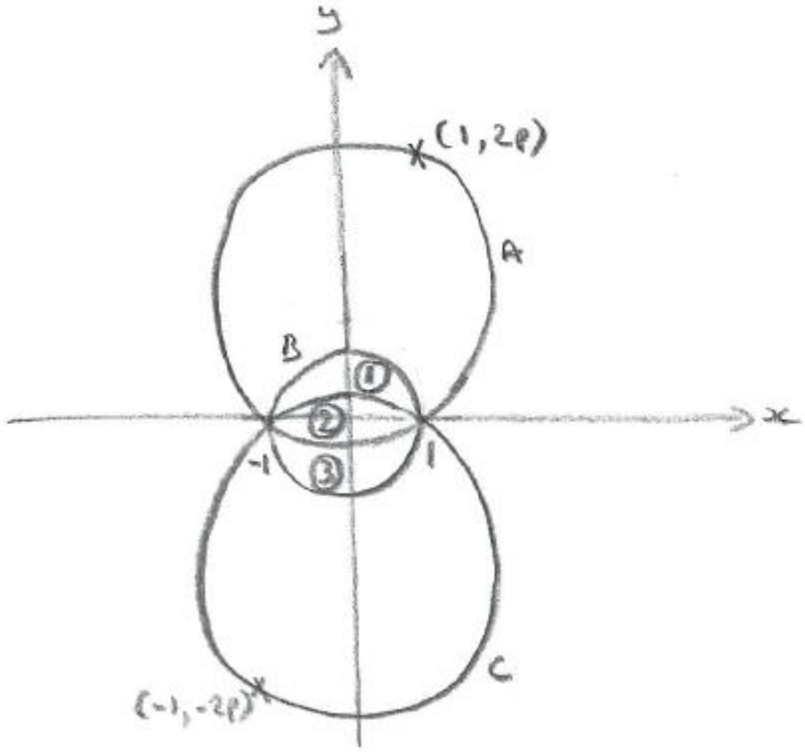
and $A_1 =$ half of area of B $+ A_3$

$= \frac{1}{2}\pi(1)^2 + \frac{\pi}{2} - 1 = \pi - 1$

Hence the lopsidedness of circle A is $1 - \frac{A_1}{A_2} = 1 - \frac{\pi-1}{2\pi}$

$= \frac{1}{2} + \frac{1}{2\pi}$ or $\frac{\pi+1}{2\pi}$

(iv) The circumference of circle A will still be outside that of circle B, above the y-axis, as A has to pass through $(1, 2p)$. By symmetry, as circle C passes through $(-1, -2p)$, it is the reflection of A in the x-axis. See diagram below: regions 1 & 3 are equal.



The lopsidedness of B is the ratio of the largest region to the whole circle.

To get a feel for what is going on, we can consider two extreme cases: (a) big gap between A and B, and (b) small gap between A and B.

For case (a), region 1 (or 3) is the largest, and to minimise the ratio, we want region 1 (and 3) to equal region 2.

For case (b), region 2 is the largest, and to minimise the ratio, we want region 2 to equal region 1 (and 3) - again.

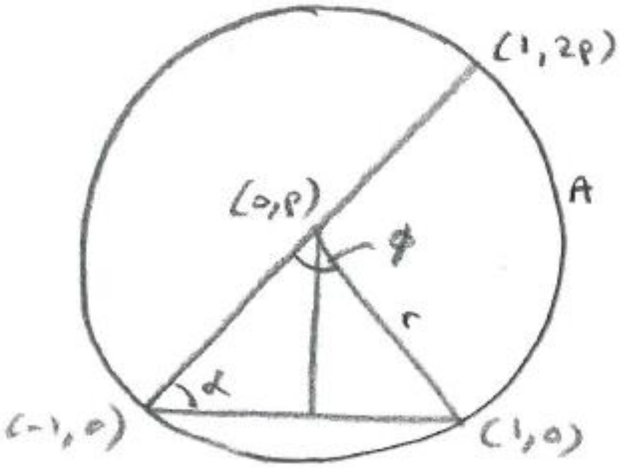
In other words, in all circumstances, we want the 3 regions to be equal.

This will be the case when region 2 has area of $\frac{1}{3} \times$ area of B; ie $\frac{\pi}{3}$

Similarly to part (ii), the area of region 2

$= 2 \times \frac{1}{2} r^2(\phi - sin\phi)$ (see diagram below)

and hence $r^2(\phi - sin\phi) = \frac{\pi}{3}$



First of all, $r^2 = p^2 + 1$ (from the diagram),

so that $(p^2 + 1)\ (\phi - sin\phi) = \frac{\pi}{3}$ (*)

The presence of $\frac{\pi}{6}$ in the given result suggests that we should be looking at $\frac{\phi}{2}$, and we want $\frac{\phi}{2} = tan^{-1}(\frac{1}{p})$

From the diagram, $\tan\left(\frac{\phi}{2}\right) = \frac{1}{p}$, as required.

Also, $\frac{sin\phi}{2} = \frac{sin\alpha}{r} = \frac{(p/r)}{r} = \frac{p}{p^2+1}$

Thus $(*) \Rightarrow (p^2 + 1)tan^{-1}\left(\frac{1}{p}\right) - p = \frac{\pi}{6}$, as required.