2015 MAT Paper - Q3 (4 pages; 24/9/20)

(i) [We can of course look ahead in the question, to see what sort of functions might be used.]

$f(x) = x,\ \ g(x) = x + \frac{1}{100}$

(ii) $|f(x) - g(x)| = \frac{1}{400} sin(4x^2)$ when $0 \le x \le \frac{1}{2}$

(as $4\left(\frac{1}{2}\right)^2 = 1 < \pi$, so that $sin(4x^2) > 0$

and $\frac{1}{400} sin(4x^2) \le \frac{1}{400}\sin(1) < \frac{1}{400}\sin\left(\frac{\pi}{3}\right) = \frac{1}{400}\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{1}{320}\left(\frac{320\sqrt{3}}{800}\right) = \frac{1}{320}\left(\frac{4\sqrt{3}}{10}\right)$

$< \frac{1}{320}\left(\frac{4\times 1.8}{10}\right) = \frac{1}{320}(0.72) < \frac{1}{320}$

(iii) $g(x) = 1 + \int_0^x 1 + t + \frac{t^2}{2} + \frac{t^3}{6}\ dt$

$= 1 + \left[t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24}\right]_0^x$

$= 1 + \ x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}$

Then $|f(x) - g(x)| = \frac{x^4}{24} \le \frac{1}{16(24)} = \frac{1}{384} < \frac{1}{320}$ when $0 \le x \le \frac{1}{2}$

(iv) RHS $= g(x) - f(x) + \int_0^x \big(h(t) - f(t)\big)dt$

$= 1 + \int_0^x f(t)\ dt - f(x) + \int_0^x h(t)dt - \int_0^x f(t)dt$

$= 1 + \int_0^x h(t)dt - f(x) = h(x) - f(x) = \text{LHS}$

(v) [This is a stand-alone result; ie not needing to be derived from the earlier results.]

Consider the area under the graph of $h(t) - f(t)$, between 0 & x.

Assume for the moment that the graph lies above the t-axis.

The maximum height of the function is $h(x_0) - f(x_0)$, and the area under the graph is no greater than the rectangle with base x and height $h(x_0) - f(x_0)$.

As $x \leq \frac{1}{2}$, the rectangle has area $\leq \frac{1}{2}(h(x_0) - f(x_0))$.

As the integral would have a smaller value if part of the graph were to lie below the t-axis,

$\int_0^x \big(h(t) - f(t)\big)dt \leq \frac{1}{2}(h(x_0) - f(x_0))$ whenever $0 \leq x \leq \frac{1}{2}$

(vi) Result to prove: $|f(x) - h(x)| \leq \frac{1}{100}$ for $0 \leq x \leq \frac{1}{2}$

or, as we are told that $f(x) \leq h(x)$,

and if we set $k(x) = h(x) - f(x)$,

result to prove is $k(x) \leq \frac{1}{100}$

From (iv), using (iii) & (v),

$k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$ (A)

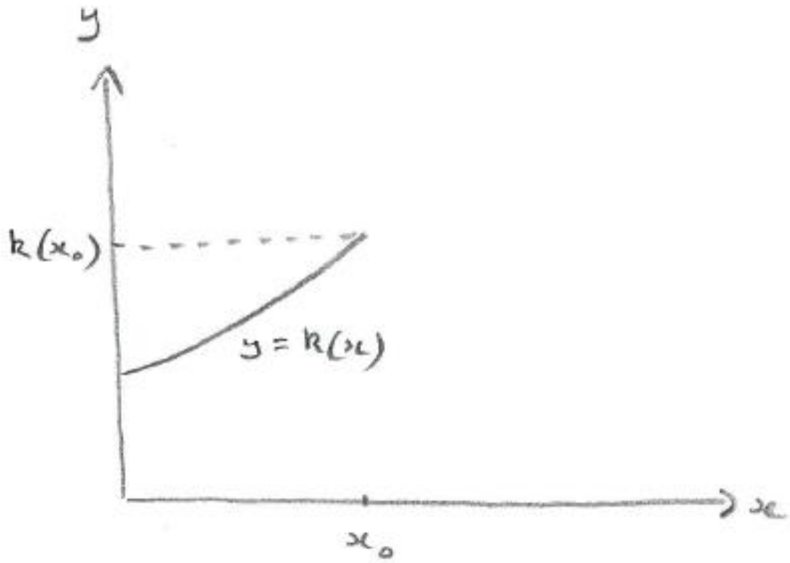
(since, from the working of (iii), $g(x) > f(x)$, so that

$g(x) - f(x) = |g(x) - f(x)| \leq \frac{1}{320}$)

Also, $k(x) \leq k(x_0)$ (B)

[At first sight, this doesn't look promising, as the inequalities in (A) & (B) seem to be in unfavourable directions:

$k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0) \Rightarrow k(x_0) \geq 2k(x) - \frac{1}{160}$, but this can't be usefully combined with (B).



However, if we consider a simple example of a graph of $k(x)$, with an upper limit of $k(x_0)$ [see diagram], and note that $k(x)$ can't be above

$\frac{1}{320} + \frac{1}{2}k(x_0)$, then we see that this doesn't work if $k(x_0)$ is very large relative to $\frac{1}{320}$, but that it can do if $k(x_0)$ is small enough relative to $\frac{1}{320}$

(in general, a useful device is to consider extreme situations)

So we need to be looking for an upper limit for $k(x_0)$.

From (A), $k(x) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$ whenever $0 \leq x \leq \frac{1}{2}$

In particular, $k(x_0) \leq \frac{1}{320} + \frac{1}{2}k(x_0)$, so that $\frac{1}{2}k(x_0) \leq \frac{1}{320}$ and $k(x_0) \leq \frac{1}{160}$

Then $k(x) \leq k(x_0) \leq \frac{1}{160} < \frac{1}{100}$,

and $k(x) \leq \frac{1}{100}$, as required.