2015 MAT Paper - Q2 (2 pages; 24/9/20)

(i) The expression simplifies to $a^{n+1} - b^{n+1}$. This is a standard result, valid for all integer n [viewed as a function of a , $f(a)$ say, $f(b) = b^{n+1} - b^{n+1} = 0$ for all n, so that $(a - b)$ is a factor]. But the companion result

$a^{n+1} + b^{n+1} = (a + b)(a^n - a^{n-1}b + a^{n-2}b^2 - \cdots - ab^{n-1} + b^n)$

[note that the signs alternate]

is only valid for even n, since (viewed as a function of a)

$f(-b) = (-b)^{n+1} + b^{n+1} = 0$ (and $(a + b)$ is a factor) only if n is even.

(ii) Suppose that $p = n^2 - 1$, where p is prime.

Then $p = (n - 1)(n + 1)$. As p is prime, we must have

$n - 1 = 1$ & $n + 1 = p$; ie $n = 2$ & $p = 3$

So there are no other prime numbers with this property.

(iii) Let $p = n^3 + 1 = (n + 1)(n^2 - n + 1)$, where p is prime and $n > 0$.

As p is prime and $n > 0$, we must have

$n + 1 = p$ & $n^2 - n + 1 = 1$

Thus $n(n - 1) = 0$, so that $n = 1$ and $p = 2$

ie the only prime number with this property is 2

(iv) The result in (i) can't be applied with $a = 3$ & $b = 2$, as this just gives a factor of $a - b = 1$

However, $3^{2015} - 2^{2015}$ can be arranged as $(3^{403})^5 - (2^{403})^5$

[or as $(3^5)^{403} - (2^5)^{403}$, or even as $(3^{31})^{13\times5} - (2^{31})^{13\times5}$ etc]

so that $3^{403} - 2^{403}$ is a factor.

Thus $3^{2015} - 2^{2015}$ isn't a prime number.

(v) [It is natural to wonder if one of the previous parts is relevant. The official solution manages to use (i) in its alternative approach.]

By way of exploration, we can consider

$(k+1)^3 = k^3 + 3k^2 + 3k + 1 > k^3 + 2k^2 + 2k + 1$ (for $k > 0$)

and so the solution is very simple: the given expression lies between k^3 & $(k+1)^3$, and hence there is no positive integer k for which $k^3 + 2k^2 + 2k + 1$ is a cube.

[Note that there are a couple of errors in the official solution: In the first line it says "Note that $k^3 < k^3 + 2k^2 + 2k$". Presumably "Note that $k^3 < k^3 + 2k^2 + 2k + 1$" was intended.

Then in the 4th line of the alternative approach it says:

"So $n \geq k + 1$, so $n^2 + nk + k^2 \leq 3k^2 + 3k + 1$", where the $\leq$ should be a $\geq$

(Incidentally, this is a good example of why it's important to include your working: Had the statement read:

"So $n \geq k + 1$, so $n^2 + nk + k^2 \leq (k+1)^2 + (k+1)k + k^2 = 3k^2 + 3k + 1$", then the error would have been much clearer, and the rest of the argument might have been considered - or the error may have been spotted by the candidate.)]