2015 MAT Paper - Multiple Choice (9 pages; 1/11/20)

Q1/A

Introduction

From the official solutions it is apparent that you are not expected to establish whether each statement is true or false; you only need to deduce which of (a)-(e) must be the correct answer.

'Whole number' should just mean integer (positive, negative or zero), and note the reference to 'positive whole number' in Q1/B (reassuring us that the examiners are not limiting whole numbers to positive values - in case we didn't trust them!)

Solution

The final answer takes the form $4(n+1)^2-3$, where n is an integer. This reveals that I is true, so that we can eliminate (c).

Based on starting values of $1, 2, 3, \ldots$, we obtain final answers of: $1, 13, 33, 61$

II, III & IV are therefore false,

and so the only possible **answer is (e).**

[V can be investigated as follows:

We want to show that $4(n+1)^2-3$ is not $\equiv 2 \; mod \; 3$

or $4(n+1)^2$ is not $\equiv 2 \; mod \; 3$

ie either $4(n+1)^2 \equiv 0 \; or \; 1 \; mod \; 3$

Case 1: $\boldsymbol{n \equiv 2 \; mod \; 3} \Rightarrow 4(n+1)^2 \equiv 0$

Case 2: $\boldsymbol{n \equiv 0 \; mod \; 3} \Rightarrow 4(n+1)^2 = n(4n+8)+4 \equiv 4 \equiv 1$

Case 3: $\boldsymbol{n \equiv 1 \; mod \; 3} \Rightarrow N = n-1 \equiv 0$, so that

$4(n+1)^2 = 4(N+2)^2 = N(4N+16) + 16 \equiv 16 \equiv 1$

So, in all cases, $4(n+1)^2 \equiv 0 \ or \ 1$, as required.]

Q1/B

Introduction

We could do sketches for small values of n, to get a feel for the problem (and possibly rule out some of the answers). This might be time-consuming though. Alternatively, an algebraic approach may be possible.

Solution

The number of intersections is the number of distinct roots of

$f(x) = f'(x)$; ie $(x+a)^n - n(x+a)^{n-1} = 0$;

or $(x+a)^{n-1}\{(x+a) - n\} = 0$

Thus there will always be exactly 2 distinct roots: $x = -a$ and

$x = n - a$ (as $n \neq 0$). **So the answer is (b)**, as 2 is an even number.

[The official solution isn't correct. $f(x)$ is an even function when $f(-x) = f(x)$, and an odd function when $f(-x) = -f(x)$, so $f(x) = (x+a)^n$ will be neither even nor odd, unless $a = 0$. Presumably, they were referring to the power of $x + a$.

Also, they say that the 2nd root will occur for a positive x, but this won't be the case if $a > n$.]

Q1/C

Solution

Considering I: $sin\left(\frac{\pi}{2}+x\right)=cos(\frac{\pi}{2}-\left[\frac{\pi}{2}+x\right])$

$=cos(-x)\not\equiv cos(\frac{\pi}{2}-x)$

so I isn't true (for all real values of x),

and therefore (a) & (b) can be eliminated.

[Note now that a further two options will be eliminated if any one of I, II or IV can be shown to be false. And III looks like an easy one to investigate.]

Considering III: $sin\left(x+\frac{3\pi}{2}\right)=cos(\frac{\pi}{2}-\left[x+\frac{3\pi}{2}\right])$

$=\cos(-\pi-x))\equiv cos(\pi-x)$

so III is true (for all real values of x) [not the hoped for result!]

and therefore (e) can be eliminated.

So the answer must be (c) or (d).

Now, $sinxcosx\leq\frac{1}{4}\Leftrightarrow sin2x\leq\frac{1}{2}$ [although compound angle formulae are obviously not in the MAT syllabus], so that IV is false.

Hence the answer must be (c).

[Considering II: $2+2sinx-cos^2x=2+2sinx-(1-sin^2x)$

$=sin^2x+2sinx+1=(sinx+1)^2\geq 0$, so that II is true.]

[The official sol'n contains some errors. It says that "III translates the $sinx$ graph by $\frac{3\pi}{2}$ along the x-axis, giving $cosx$". This should read

"III translates the $sinx$ graph by $\frac{3\pi}{2}$ along the x-axis (in the negative direction), giving $-cosx$" (perhaps they were confusing it with $sin\left(x - \frac{3\pi}{2}\right)$, which **is** $cosx$). Also,the statement "$cos(\pi - x)$ translates the $cosx$ graph by π and reflects in the x-axis, giving $cosx$" should read "$cos(\pi - x)$ $[= \cos(x - \pi)]$ translates the $cosx$ graph by π, giving $-cosx$".]

Q1/D

Solution

$f(x) = \int_0^1 (xt)^2 dt = x^2 \int_0^1 t^2 dt = x^2 [\frac{1}{3}t^3]_0^1 = \frac{1}{3}x^2$

$g(x) = \int_0^x t^2 dt = \left[\frac{1}{3}t^3\right]_0^x = \frac{1}{3}x^3$

$f\left(g(A)\right) = \frac{1}{3}(\frac{1}{3}A^3)^2 = \frac{1}{27}A^6$

$g\left(f(A)\right) = \frac{1}{3}(\frac{1}{3}A^2)^3 = \frac{1}{81}A^6$

So the answer is (b).

Q1/E

Solution

With $u = 2cos(2x) + 2, 0 \leq x \leq 2\pi \Rightarrow 2(-1) + 2 \leq u \leq 2(1) + 2$

ie $0 \leq u \leq 4$

Then $sinu = 0 \Rightarrow u = 0 \; or \; \pi$

[Note: we don't need to consider $u = 2\pi$ (as the Official Sol'n does), as $2\pi > 4$]

$\Rightarrow \cos(2x) = -1 \; or \; \frac{\pi - 2}{2} = \frac{\pi}{2} - 1$

Now making the substitution $w = 2x$,

$0 \le x \le 2\pi \Rightarrow 0 \le w \le 4\pi$

Referring to the graph of $cosw$,

$cosw = -1$ has 2 solutions (for w), and $cosw = \frac{\pi}{2} - 1$ has 4 solutions; making 6 solutions in total.

As $x = \frac{w}{2}$, there are also 6 solutions for x.

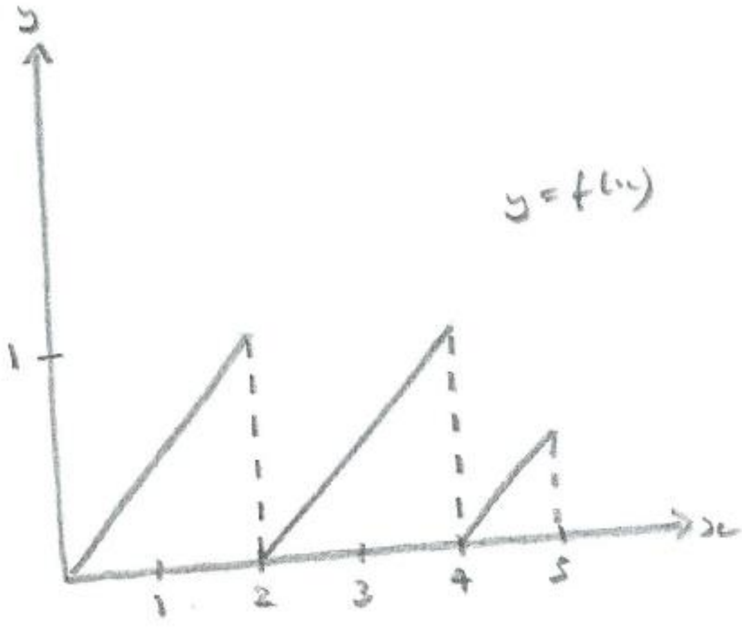
So the answer is (d).

Q1/F

Solution

The graph of $f(x)$ is shown below,

and $\int_0^5 f(x)\,dx = \frac{1}{2}(2)(1) + \frac{1}{2}(2)(1) + \frac{1}{2}(1)\left(\frac{1}{2}\right) = 2\frac{1}{4} = \frac{9}{4}$



The trapezium rule estimate with 2 strips is

$\frac{1}{2}\left(\frac{5}{2}\right)\left(f(0) + 2f\left(\frac{5}{2}\right) + f(5)\right) = \frac{5}{4}\left(0 + (2)\frac{1}{4} + \frac{1}{2}\right) = \frac{5}{4}$;

ie an underestimate

The trapezium rule estimate with 3 strips is

$\frac{1}{2}\left(\frac{5}{3}\right)\left(f(0) + 2(f\left(\frac{5}{3}\right) + f\left(\frac{10}{3}\right)) + f(5)\right)$;

$= \frac{5}{6}\left(0 + 2\left(\frac{5}{6} + \frac{2}{3}\right) + \frac{1}{2}\right) = \frac{35}{12}$

ie an overestimate

So (b) is the correct answer.

Q1/G

Introduction

The relation $cos\theta = sin(\frac{\pi}{2} - \theta)$ can be useful, as an algebraic way of dealing with $-cos\theta$ (although referring to the graph of $cos\theta$ is quicker).

Solution

$cos^2x = cos^2y \Rightarrow cosx = \pm cosy$

Then $cosx = cosy \Rightarrow y = x + 2n\pi$ or $y = (2\pi - x) + 2n\pi$

ie $y = 2n\pi \pm x$

[Alternatively, the two base angles can be taken as x & $-x$]

And $cosx = -cosy \Rightarrow cosy = -sin\left(\frac{\pi}{2} - x\right) = sin(x - \frac{\pi}{2})$

$= cos\left(\frac{\pi}{2} - \left(x - \frac{\pi}{2}\right)\right) = cos(\pi - x)$

[or just refer to the graph of $cosx$, to see that

$-cosx = cos(\pi - x)]$

Thus $y = \pi - x + 2n\pi = (2n+1)\pi - x$,

or $y = 2\pi - (\pi - x) + 2n\pi = (2n+1)\pi + x$

ie $y = (2n+1)\pi \ \pm x$

Thus the possible solutions are:

$y = 2n\pi \ \pm x$ and $y = (2n+1)\pi \ \pm x$;

ie straight lines with gradients of ± 1, with y-intercepts of even and odd multiples of π,

so that **(c) is the correct answer.**

Q1/H

Solution

$log_{x^2+2}(4 - 5x^2 - 6x^3) = 2 \Rightarrow$

$4 - 5x^2 - 6x^3 = (x^2+2)^2 = x^4 + 4x^2 + 4$

$\Rightarrow x^4 + 6x^3 + 9x^2 = 0$

$\Rightarrow x^2(x^2 + 6x + 9) = 0$

$\Rightarrow x^2(x+3)^2 = 0$

Thus there are two distinct sol'ns,

and so the answer is (c).

Q1/I

Solution

As well as noting the points of intersection of the 3 curves, we can also consider their gradients in appropriate regions.

If $f(x) = x^3, g(x) = x^4$ & $h(x) = x^5$,

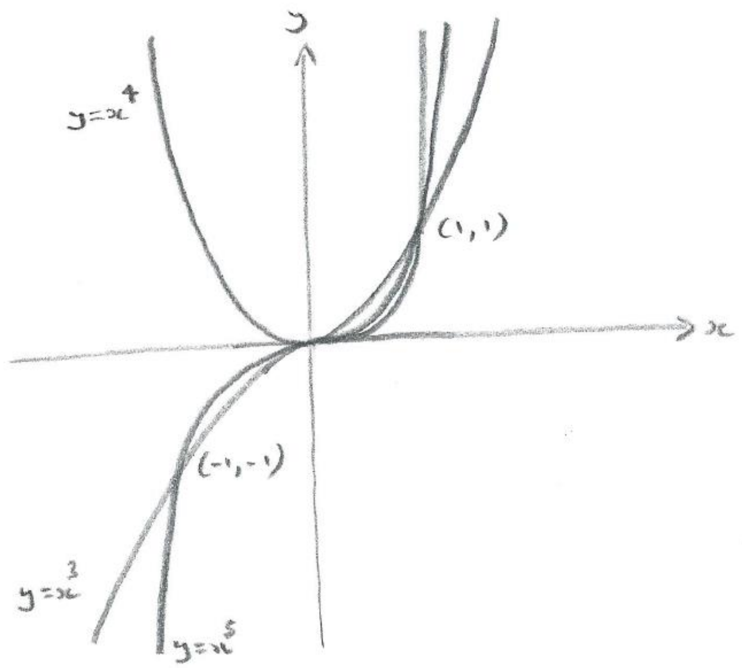
then $f'(x) = 3x^2, g'(x) = 4x^3$ & $h'(x) = 5x^4$,

so that, for $x > 0$ for example, $g'(x) > f'(x) \Rightarrow x > \frac{3}{4}$

and $h'(x) > g'(x) \Rightarrow x > \frac{4}{5}$;

ie $g(x)$ is initially less steep than $f(x)$, but the curves cross at $x = 1$; and similarly for $h(x)$ & $g(x)$

There are 9 regions in the diagram below.



So the answer is (d).

Q1/J

Introduction

If we want to compare expressions x & y, we can, for example, compare x^2 & y^2 instead, or see whether $\frac{x}{y} > 1$, or $x - y > 0$. The last option is often the easiest. For awkward expressions such as $\frac{log_2 30}{log_3 85}$, an approximation may be good enough. It's certainly best to leave it until last, when we know which of the other expressions it needs to be compared with.

The official solutions for this type of question often give the impression that the correct answer is already known. In practice, you may need to go down some blind alleys in arriving at the solution.

Solution

(a) and (b) are quickest to deal with first:

$\frac{\sqrt{7}}{2} > \frac{5}{4} \Leftrightarrow \frac{7}{4} > \frac{25}{16} \Leftrightarrow \frac{28}{16} - \frac{25}{16} > 0$; so (a) > (b)

(e) is probably the next simplest; comparing $(a)^2$ & $(e)^2$:

$\frac{7}{4} - \frac{(1+2\sqrt{6}+6)}{9} = \frac{63-28-8\sqrt{6}}{36} > \frac{35-8(3)}{36} > 0$; so (a) > (e)

Next, we can consider $(c)^2 \div (a)^2$:

Thus $\frac{\left(\frac{10!}{9(6!)^2}\right)}{\left(\frac{7}{4}\right)} = \frac{(10)(9)(8)(7)(4)}{(9)(6!)(7)} = \frac{(10)(8)}{(6)(5)(3)(2)} = \frac{8}{18} < 1$,

so that (a) > (c).

Finally we need to either compare (d) with (a) directly, or perhaps show that it less than (b) [being simpler than (a)].

Considering the powers of 2 and 3, we see that $log_2 30$ is close to 5, whilst $log_3 85$ is close to 4. [In general, we can only expect to use fairly good approximations to demonstrate inequalities.]

So $\frac{log_2 30}{log_3 85} < \frac{5}{4}$, and $(d) < (b)$. Thus (a) is the largest one.

Answer: (a)