2014 MAT Paper - Q5 (2 pages; 24/9/20)

Solution

(i) AAA, AAB, ABA, ABB, ABC

(ii) $c_{n,1} = 1$: The letters are all A.

$c_{n,n} = 1$: The n letters have to be in alphabetical order.

(iii) Case 1: The final position is different from the others.

There are $c_{n-1,k-1}$ possibilities for the other positions, and just one for the final position (as it has to be a new letter);

ie $c_{n-1,k-1}(1)$ possibilities

Case 2: The final position is the same as one of the others.

There are $c_{n-1,k}$ possibilities for the other positions, and k possibilities for the final position;

ie $c_{n-1,k}(k)$ possibilities;

giving a total of $c_{n-1,k-1}(1) + c_{n-1,k}(k) = kc_{n-1,k} + c_{n-1,k-1}$

(iv) $r_{\text{n}} = \sum_{k=1}^{\text{n}} c_{\text{n},k}$

$r_4 = \sum_{k=1}^{4} c_{4,k} = c_{4,1} + c_{4,2} + c_{4,3} + c_{4,4}$

$= 1 + (2c_{3,2} + c_{3,1}) + (3c_{3,3} + c_{3,2}) + 1$

$= 2 + c_{3,1} + 3c_{3,2} + 3c_{3,3}$

$= 2 + 1 + 3(2c_{2,2} + c_{2,1}) + 3$

$= 6 + 3(2 + 1)$

$= 15$

(v) [The official sol'ns gives a quick 'lateral thinking' way of answering this question; not using (iii). The form of words "Give a formula ... Justify your answer" perhaps suggests that (iii) needn't be used.]

From (iii), $c_{n,2} = 2c_{n-1,2} + c_{n-1,1}$

$= 2\left(2c_{n-2,2} + c_{n-2,1}\right) + 1 \quad \text{for } n > 2$

$= 4c_{n-2,2} + 2 + 1$

$= 4(2c_{n-3,2} + c_{n-3,1}) + 2 + 1 \quad (\text{for } n > 3)$

$= 8c_{n-3,2} + 4 + 2 + 1$

$\ldots = 2^{n-2}c_{n-(n-2),2} + 2^{n-3} + \cdots + 2 + 1 \quad \text{for } n > 2$

$= 2^{n-2} + \cdots + 2 + 1$

$= \frac{2^{n-1}-1}{2-1}$

$= 2^{n-1} - 1$

And for $n = 2,\ 2^{n-1} - 1 = 1 = c_{2,2}$

Thus $c_{\text{n},2} = 2^{n-1} - 1$ for all $n \geq 2$