2014 MAT Paper - Q4 (2 pages; 25/11/20)

(i) Area $= \frac{1}{2}AB.BCsin(\angle ABC) = \frac{1}{2}(1)(1)sin(\pi - 2\alpha)$,

As ΔABC is isosceles, and so $\angle ACB = \alpha$

So area $= \frac{1}{2}sin(2\alpha)$, as required.

(ii) When $\beta = \alpha, D = C$ and $F = 1$

When $\beta = \frac{\pi}{2}, D = A$ and $F = 0$

And as β increases, F reduces, with F being a continuous function of β.

So there will be a $1 - 1$ correspondence between the values of β and F, and hence there is a unique value of β such that $F = k$.

[The official sol'n considers the situation when $\beta = 0$, but this isn't allowed by the assumption that $0 < \alpha \leq \beta$.]

(iii) $F = \frac{1}{2}$ when $\angle AXB = \frac{\pi}{2}$,

Then $\angle ABD = \angle ABX = \frac{\pi}{2} - \alpha$,

so that $2\beta + \left(\frac{\pi}{2} - \alpha\right) = \pi$ (as ΔABD is isosceles),

and hence $\beta = \frac{1}{2}\left(\frac{\pi}{2} + \alpha\right) = \frac{\pi}{4} + \frac{\alpha}{2}$

(iv) To find the area of ΔABX:

$\angle ABX = \angle ABD = \pi - 2\beta$, so that $\angle AXB = \pi - \alpha - (\pi - 2\beta) = 2\beta - \alpha$

And $\frac{XB}{sin\alpha} = \frac{AB}{\sin(\angle AXB)} = \frac{1}{\sin(2\beta - \alpha)}$,

So that $XB = \frac{sin\alpha}{\sin(2\beta - \alpha)}$

And area of $\Delta ABX = \frac{1}{2}\, AB.XBsin(\angle ABX)$

$= \frac{1}{2}(\frac{sin\alpha}{\sin(2\beta-\alpha)})\sin(\pi - 2\beta)$

$= \frac{sin\alpha \sin(2\beta)}{2\sin(2\beta-\alpha)}$ (1)

Then $F = \frac{sin\alpha \sin(2\beta)}{2\sin(2\beta-\alpha)[\frac{1}{2}\sin(2\alpha)]} = \frac{\sin(2\beta) sin\alpha}{\sin(2\beta-\alpha)\sin(2\alpha)]}$, as required.

(v) With $\beta < \alpha$, the diagram appears as before, but with α & β swapped, as well as C & D.

Then the area of ΔABX is obtained by swapping α & β in (1) of (iv), to give $\frac{sin\beta \sin(2\alpha)}{2\sin(2\alpha-\beta)}$

Noting that $\angle CAB$ is still defined to be α, the area of ΔABC still equals $\frac{1}{2} sin(2\alpha)$,

And so $F = \frac{sin\beta \sin(2\alpha)}{2\sin(2\alpha-\beta).\frac{1}{2}sin(2\alpha)} = \frac{sin\beta}{\sin(2\alpha-\beta)}$