2014 MAT Paper - Q3 (2 pages; 25/11/20)

(i) From (A), $f(x+0) = f(x)f(0)$,

So that $f(0) = \frac{f(x)}{f(x)}$ (as $f(x) \neq 0$), and thus $f(0) = 1$, as required.

(ii) As $f'(x) = f(x)$, $\int f(x)dx = f(x) + c$,

so that $I = [f(x)]_0^1 = f(1) - f(0) = a - 1$, as required.

(iii) ['steps' may be a typo, as 'strips' is more common]

$I_n = \frac{1}{2}(\frac{1}{n})(f(0) + f(1) + 2\left[f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n-1}{n}\right)\right])$

Then, as $f(x+y) = f(x)f(y)$,

$f\left(\frac{2}{n}\right) = f\left(\frac{1}{n} + \frac{1}{n}\right) = f\left(\frac{1}{n}\right)f\left(\frac{1}{n}\right) = [f\left(\frac{1}{n}\right)]^2$,

And $f\left(\frac{3}{n}\right) = f\left(\frac{2}{n} + \frac{1}{n}\right) = f\left(\frac{2}{n}\right)f\left(\frac{1}{n}\right) = [f\left(\frac{1}{n}\right)]^3$ and so on.

Then, with $b = f\left(\frac{1}{n}\right)$, $I_n = \frac{1}{2n}(1 + a + 2[b + b^2 + \cdots + b^{n-1}])$

$= \frac{1}{2n}[1 + a + \frac{2b(b^{n-1}-1)}{b-1}]$

And $b^n = f\left(\frac{n}{n}\right) = a$,

So that $I_n = \frac{1}{2n}\left(\frac{1}{b-1}\right)[(1+a)(b-1) + 2a - 2b]$

$= \frac{1}{2n}\left(\frac{1}{b-1}\right)[b - 1 + ab - a + 2a - 2b]$

$= \frac{1}{2n}\left(\frac{1}{b-1}\right)[-b - 1 + ab + a]$

$= \frac{1}{2n}\left(\frac{1}{b-1}\right)(b+1)(-1+a)$

$= \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1)$, as required.

(iv) rtp [result to prove]: $a \leq \left(1+\frac{2}{2n-1}\right)^n$

Equivalently, $b = a^{\frac{1}{n}} \leq 1 + \frac{2}{2n-1}$ (*)

Now, $I_n \geq I \Rightarrow \frac{1}{2n}\left(\frac{b+1}{b-1}\right)(a-1) \geq a-1$,

So that $\frac{1}{2n}\left(\frac{b+1}{b-1}\right) \geq 1$

$\Leftrightarrow b+1 \geq 2n(b-1)$

$\Leftrightarrow b(1-2n) \geq -2n-1$

$\Leftrightarrow b \leq \frac{-(2n+1)}{1-2n} = \frac{2n+1}{2n-1}$, as $1-2n < 0$

$\Leftrightarrow b \leq 1 + \frac{2}{2n-1}$, which is (*)