2014 MAT Paper - Q2 (2 pages; 18/10/20)

Solution

(i) If $x = 1$ is a sol'n of the cubic, then $1 + 2b - a^2 - b^2 = 0$,

so that $b^2 - 2b - 1 = -a^2 \leq 0$ (A)

The graph of $y = f(b) = b^2 - 2b - 1$ crosses the b-axis when

$b = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}$,

so that $(A) \Rightarrow 1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}$,

as $y = f(b)$ is a u-shaped quadratic.

(ii) Let $g(x) = x^3 + 2bx^2 - a^2x - b^2$

For $x = 1$ to be a repeated root, there must be a turning point at $x = 1$.

$g'(x) = 3x^2 + 4bx - a^2$

Then $g'(1) = 0 \Rightarrow 3 + 4b - a^2 = 0$

Also, $b^2 - 2b - 1 = -a^2$, from (A), so that

$3 + 4b = -(b^2 - 2b - 1)$,

and hence $b^2 + 2b + 2 = 0$ (B)

But $b^2 + 2b + 2 = (b + 1)^2 + 1 > 0$,

so that there are no sol'ns to (B).

(iii) **1st part**

Suppose that $x^3 + 2bx^2 - a^2x - b^2 = (x - c)^2(x - 1)$.

$= (x^2 - 2cx + c^2)(x - 1)$

Equating coeffs of x^2: $2b = -1 - 2c$

Equating constant terms: $-b^2 = -c^2$

Hence $c = \pm b$, and $2b = -1 \mp 2b$.

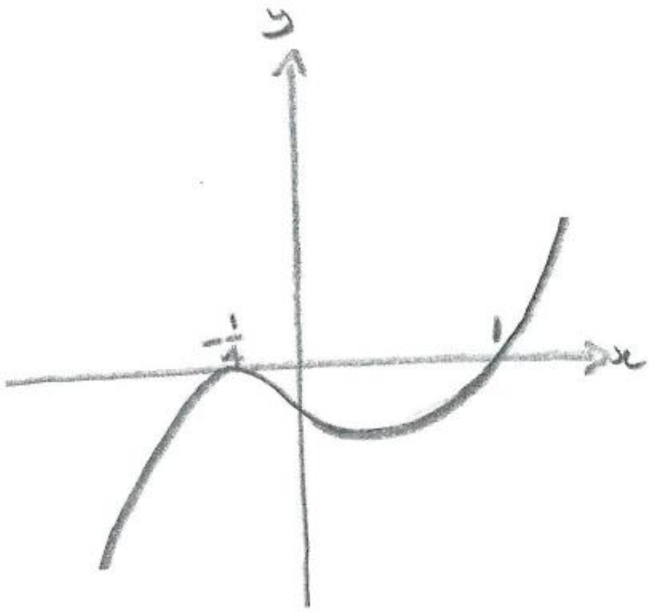
When $c = b$, $2b = -1 - 2b$, so that $4b = -1$ and $b = -\frac{1}{4}$

When $c = -b$, $2b = -1 + 2b$, and there is no sol'n.

2nd part

As $c = b$, $g(x) = \left(x + \frac{1}{4}\right)^2 (x - 1)$.

From the shape of the cubic (see diagram below), it has a maximum at its repeated root.



[Strictly speaking, we need to check that a solution exists for a:
Equating coeffs of x in the 1st part gives

$-a^2 = c^2 + 2c = \frac{1}{16} - \frac{1}{2} < 0$, so that a solution does exist for a.]