2014 MAT Paper - Multiple Choice (10 pages; 28/10/20)

Q1/A

Solution

Writing $y = x^2, x^4 < 8x^2 + 9 \Leftrightarrow y^2 - 8y - 9 < 0$

ie $(y - 9)(y + 1) < 0 \Leftrightarrow\ -1 < y < 9$

(as $y^2 - 8y - 9$ is a u-shaped quadratic)

$\Leftrightarrow -3 < x < 3$

ie the answer is (a)

Q1/B

[Completing the square is something that is quick to do (and may shed some light on the problem).]

Let $f(x) = log_{10}(x^2 - 2x + 2) = log_{10}[(x - 1)^2 + 1]$

Then $f(1) = 0$, so that (a), (b), (c) & (d) can be eliminated.

Thus the answer is (e).

Q1/C

Solution

$y = kx^3 - (k + 1)x^2 + (2 - k)x - k$

$\Rightarrow \frac{dy}{dx} = 3kx^2 - 2(k + 1)x + 2 - k$

$\Rightarrow \frac{d^2y}{dx^2} = 6kx - 2(k + 1)$

In order for there to be a turning point that is a minimum,

$\frac{dy}{dx} = 0$ and $\frac{d^2y}{dx^2} > 0$, when $x = 1$

[This is true for a cubic, but not a general function, as $\frac{d^2y}{dx^2} > 0$ is not a necessary condition for a minimum (consider $y = x^4$, for example, when $\frac{d^2y}{dx^2} = 0$ at the minimum).]

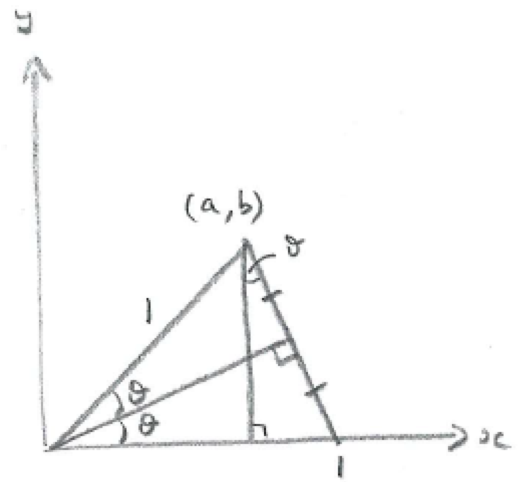
so that $3k - 2(k+1) + 2 - k = 0$, which is true for any k,

and $6k - 2(k+1) > 0$; ie $4k > 2$, or $k > \frac{1}{2}$

So (c) is the answer.

Q1/D

Method 1



From the diagram above, the x-coordinate of the reflected point is

$1(\cos(2\theta)) = cos^2\theta - sin^2\theta = 2cos^2\theta - 1 = \frac{2}{sec^2\theta} - 1$

$= \frac{2}{1+tan^2\theta} - 1 = \frac{2}{1+m^2} - 1 = \frac{2-(1+m^2)}{1+m^2} = \frac{1-m^2}{1+m^2}$,

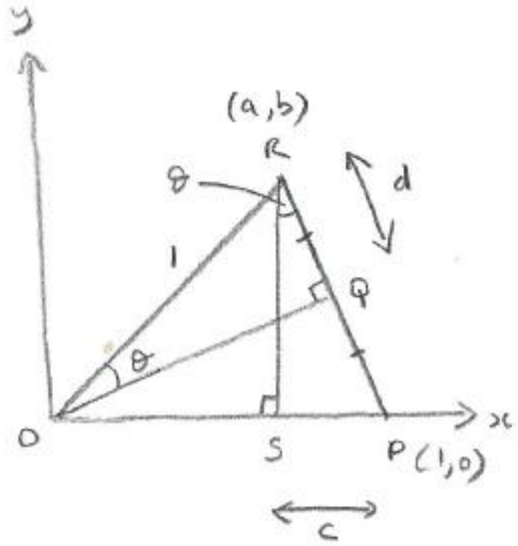
which shows that **(d) is the correct answer**.

Also, y-coordinate is $1(\sin(2\theta)) = 2sin\theta cos\theta$

$tan\theta = \frac{m}{1} \Rightarrow sin\theta = \frac{m}{\sqrt{1+m^2}}$ and $cos\theta = \frac{1}{\sqrt{1+m^2}}$ (by Pythagoras)

so that $2sin\theta cos\theta = \frac{2m}{1+m^2}$, as required.

Method 2



From the diagram above, triangle OQR gives $sin\theta = \frac{d}{1}$ and triangle PRS gives $\frac{b}{2d} = cos\theta$

Then $b = 2sin\theta cos\theta$ (1)

Also $c^2 = (2d)^2 - b^2 = (2sin\theta)^2 - 4sin^2\theta cos^2\theta$

$= 4sin^2\theta(1 - cos^2\theta) = 4sin^4\theta$

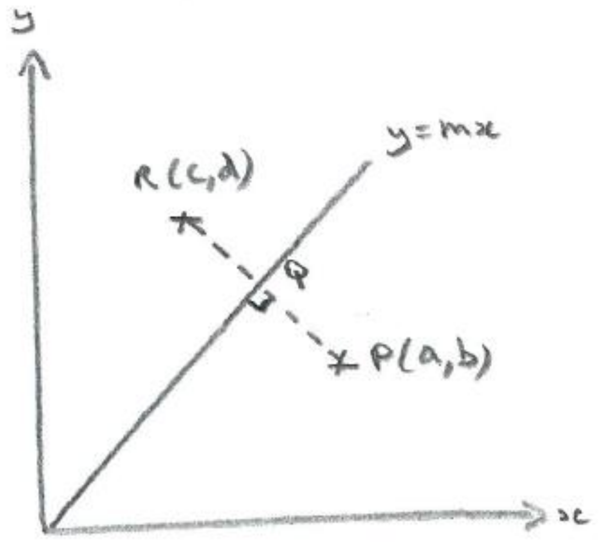
so that $c = 2sin^2\theta$

and $a = 1 - c = 1 - 2sin^2\theta = 1 - 2(1 - cos^2\theta) = 2cos^2\theta - 1$ (2)

Then from (1) & (2) we can proceed as in the previous method.

Method 3 (outside the MAT syllabus)

The reflection of the general point $P(a,b)$ in the line $y = mx$ can be obtained by the following vector method:



Referring to the diagram, let $\lambda\binom{1}{m}$ be the point Q.

Then, as $\overrightarrow{QP}$ is perpendicular to the line $y = mx$,

$\overrightarrow{QP}.\binom{1}{m} = 0$; ie $\binom{a-\lambda}{b-\lambda m}.\binom{1}{m} = 0$,

so that $a - \lambda + (b - \lambda m)m = 0$

$\Rightarrow \lambda(m^2 + 1) = a + bm$, and $\lambda = \frac{a+bm}{m^2+1}$

Then $\overrightarrow{OR} = \overrightarrow{OQ} + \overrightarrow{QR} = \lambda\binom{1}{m} + \overrightarrow{PQ}$

$= \lambda\binom{1}{m} + \binom{\lambda - a}{\lambda m - b}$

$= \binom{2\lambda - a}{2\lambda m - b}$

$$= \frac{1}{m^2+1}\begin{pmatrix} 2(a+bm) - a(m^2+1) \\ 2m(a+bm) - b(m^2+1) \end{pmatrix}$$

$$= \frac{1}{m^2+1}\begin{pmatrix} a(1-m^2) + 2bm \\ 2am + b(m^2-1) \end{pmatrix}$$

[Note that, when $m = 1$, R is (b, a).]

Then, when $a = 1$ & $b = 0$, R is $\frac{1}{m^2+1}\begin{pmatrix} 1-m^2 \\ 2m \end{pmatrix}$

So the answer is (d).

Q1/E

$(4sin^2x + 4cosx + 1)^2 = (4(1 - cos^2x) + 4cosx + 1)^2$

$= (5 + 4cosx - 4cos^2x)^{\mathbf{2}}$

$= (-4(cos^2x - cosx) + 5\ )^{\mathbf{2}}$

$= (-4\left(cosx - \frac{1}{2}\right)^2 + 6\ )^{\mathbf{2}}$

$= (6 - (2cosx - 1)^2\ )^2$ (A)

Now, $(2cosx - 1)^2$ varies from 0 to 9

When it equals 0, (A) has its maximum value of 36.

So the answer is (b).

Q1/F

Introduction

This problem can be investigated either algebraically, or graphically. In fact a combination of the two approaches is probably the quickest way of arriving at the answer.

Solution

From an algebraic point of view, any combination of the functions S and T will produce a function of the form $a \pm x$ (where a is an integer), so that t has to be odd, in order to give the $-x$ in

$F(x) = 8 - x$.

From a graphical point of view, if x represents the initial position on the x-axis, with T having the effect of a reflection in the line $x = 0$, and S that of a translation of one place to the right, then $F(x)$ can be considered to represent the final position.

If x is odd/even, $8 - x$ will be odd/even as well, and there will be an even difference between the final and initial positions in both cases, so that s must be even, as any reflections about $x = 0$ have no effect on the odd/even status.

Thus t is odd and s is even, **so that the answer is (c).**

Q1/G

Solution

$([1 + xy] + y^2)^n = (1 + xy)^n + n(1 + xy)^{n-1}y^2$

$+ \binom{n}{2}(1 + xy)^{n-2}y^4 + \cdots$

Only the term $n(1+xy)^{n-1}y^2$ will contain a power of y that is 2 greater than the power of x, and we require the term $n\binom{n-1}{3}(xy)^3y^2$ within this, so that the required coefficient is

$$n\binom{n-1}{3} = \frac{n(n-1)!}{3!(n-4)!}$$

$$= \frac{n!(4!)}{3!(n-4)!(4!)} = 4\binom{n}{4}$$

so that the answer is (d).

Alternatively, knowledge of the trinomial expansion:

$$(a+b+c)^n = \sum_{\substack{i,j,k \\ (i+j+k=n)}} \binom{n}{i,j,k} a^i b^j c^k,$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$, gives the answer straightaway,

as we require the term $\binom{n}{n-4,3,1}(1)^{n-4}(xy)^3(y^2)^1$,

so that the coefficient is

$$\binom{n}{n-4,3,1} = \frac{n!}{(n-4)!(3!)(1!)} = \frac{n!(4!)}{(n-4)!(3!)(4!)} = 4\binom{n}{4}$$

Q1/H

To find out how much has been added to $F(1)$ by the time $F(6000)$ has been reached:

Of the numbers 2-6000,

3000 are even

2000 are multiples of 3, and 1000 of these are multiples of 2

[6000, 5994, ... , 6, with $6000 = 6 + 999 \times 6$]

The numbers in the required categories are:

2 divides n but 3 does not divide n (3000-1000 $= 2000$)

3 divides n but 2 does not divide n (1000)

2 and 3 both divide n (1000)

So $F(6000) = 2000(2) + 1000(3) + 1000(4) = 11000$

so that the answer is (c).

Q1/I

Introduction

At each stage of a composite transformation, we must always be doing one of the following:

(a) replacing x with $x + a$ (where a can be negative), to give a translation of $\binom{-a}{0}$

(b) replacing x with kx (where k can be negative; eg $k = -1$ represents a reflection in the y-axis), to give a stretch of factor $1/k$ in the x-direction (ie the graph is seen to compress if $k > 1$)

(c) replacing y with $y + a$ (or, equivalently, subtracting a from the function, so that $y = f(x) \to y = f(x) - a$),), to give a translation of $\binom{0}{-a}$

(d) replacing y with ky (or , equivalently, dividing the function by k, so that $y = f(x) \to y = \frac{1}{k}f(x)$), to give a stretch of factor $1/k$ in the y-direction

Solution

$x^2 - 4x + 3 = (x - 2)^2 - 1$

$y = 2^{x^2} \to y = 2^{(x-2)^2}$ is a translation of $\binom{2}{0}$

and $y = 2^{(x-2)^2} \rightarrow y = 2^{(x-2)^2-1}$ or $y = \frac{1}{2}2^{(x-2)^2}$ is a stretch of scale factor 2 in the y direction

So the answer is (b).

Q1/J

Solution

[In the official solution, $\int_{-1}^{1} f(x)du$ should read $\int_{-1}^{1} f(x)dx$ (or $\int_{-1}^{1} f(u)du$) etc]

$\int_{-1}^{1} f(t)dt$ and $\int_{-1}^{1} f(x)dx$ can both be written as a constant, A say.

A possible line of investigation could be substituting in particular values of x (eg $x = -1$ & 1), or (more generally) $x = a$:

This yields $6 + f(a) = 2f(-a) + 3a^2A$ and

$6 + f(-a) = 2f(a) + 3a^2A$

Subtracting one from the other then gives $f(a) = f(-a)$.

As this is true for all a, we can say that $f(x) = f(-x)$.

Then, integrating both sides of the original equation from -1 to 1:

$\int_{-1}^{1} 6\,dx + \int_{-1}^{1} f(x)dx = 2\int_{-1}^{1} f(x)dx + 3A\int_{-1}^{1} x^2dx$

$\Rightarrow [6x]_{-1}^{1} + A = 2A + 3A[\frac{1}{3}x^3]_{-1}^{1}$

$\Rightarrow \big(6 - (-6)\big) = A + A(1 - (-1))$

$\Rightarrow 12 = 3A$

$\Rightarrow A = 4$

So the answer is (a).

[The official solution points out that $\int_{-1}^{1} f(-x)dx = \int_{-1}^{1} f(x)dx$, which shortens the method.]