2013 MAT Paper - Q5 (2 pages; 26/9/20)

Introduction

This question is a good illustration of the dilemmas to be faced for this style of question: (i) when to start listing possible combinations, and (ii) when to try to use an earlier result.

The idea of conditioning is very important.

Solution

(i) The numbers are: 8, 17, 26, 35, 44, 53, 62, 71, 80

so that there are 9 such numbers.

(ii) If $n = 9$ (for example), then the numbers are 9, 27, ..., 90,

giving 10 numbers.

If $n = 1$, then we just have 1 & 10.

In general, we see that there will be $n + 1$ numbers.

(iii) Conditioning on the hundreds digit (or the units) enables the result from (ii) to be used:

If the hundreds digit is 0, then the remaining digits must sum to n, and so there will be $n + 1$ suitable numbers, from (i).

If the hundreds digit is 1, then the remaining digits must sum to $n - 1$, and so there will be $(n - 1) + 1$ suitable numbers.

And so on, until the hundreds digit is $n - 1$, when the remaining digits must sum to 1, and so there will be $1 + 1$ suitable numbers.

Finally, if the hundreds digit is n, then the remaining digits must be zeros, so that there is 1 suitable number.

In total there are $(n + 1) + n + \cdots + 1 = \frac{1}{2}(n + 1)(n + 2)$ numbers.

(iv) We can apply the same method, but with $n = 8$, and allowing the hundreds digit to range from 5 to 9:

If the hundreds digit is 5, then the remaining digits must sum to 3, and so there will be $3 + 1$ suitable numbers.

And so on, until the hundreds digit is 7, when the remaining digits must sum to 1, and so there will be $1 + 1$ suitable numbers.

Finally, if the hundreds digit is 8 then the remaining digits must be zeros, so that there is 1 suitable number.

This gives a total of $4 + 3 + 2 + 1 = 10$ numbers.

(v) [Note that there can be no more than one digit of at least 5 (otherwise the wording is ambiguous: does "one digit" mean "exactly one" or "at least one"?)]

Suppose that the hundreds digit is at least 5. Then, from (iv), there are 10 suitable numbers.

By symmetry, there are also 10 suitable numbers, where the tens digit is at least 5, and a further 10 where the units digit is at least 5. This gives an overall total of 30.

(vi) [Note that we cannot simply add the result from (iii) for the different values of n, as n would have to range beyond 9 (and $n < 10$ for (iii).]

As there are 1000 numbers, there are 3×1000 digits overall (writing eg 20 as 020).

By symmetry, each of the digits 0 to 9 occurs the same number of times. So each one occurs $\frac{3000}{10} = 300$ times.

Then the required sum is $300(0 + 1 + 2 + \cdots + 9)$

$= 300\left(\frac{1}{2}\right)(9)(10) = 13500$