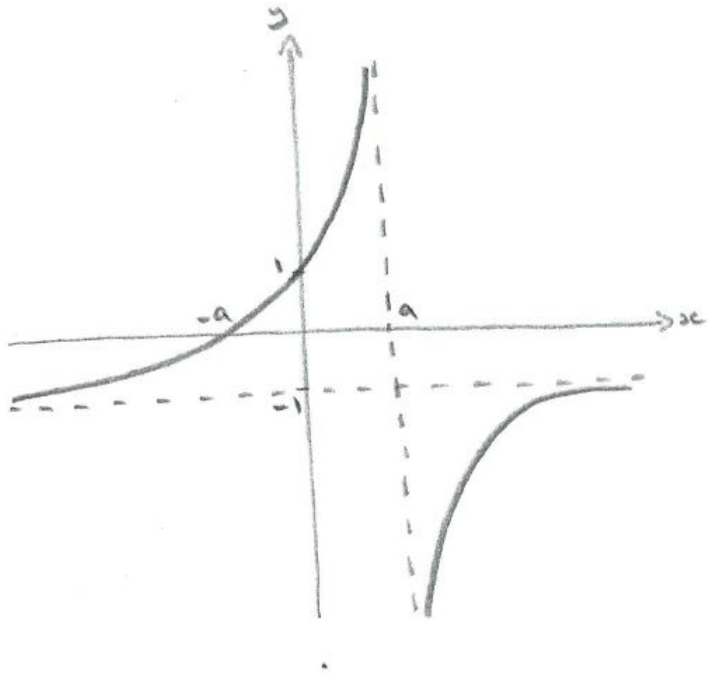
2013 MAT Paper - Q4 (3 pages; 27/11/20)

(i) First of all, without any restriction on x:

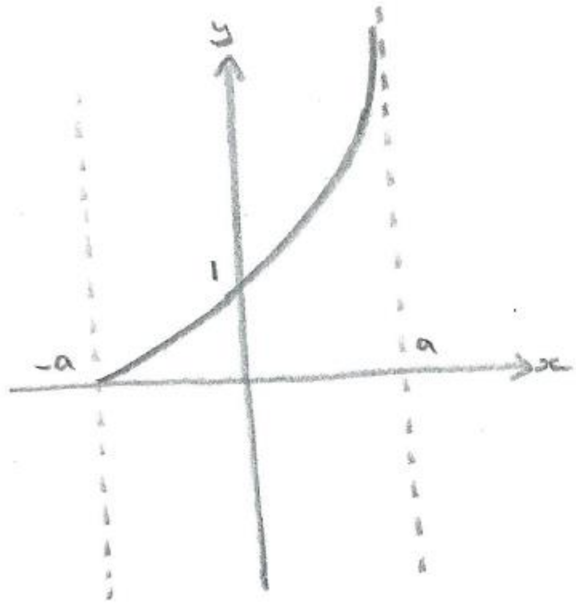
When $x = 0, y = 1$ and when $y = 0, x = -a$.

There is a vertical asymptote at $x = a$, and the behaviour of the graph of $y = \frac{a+x}{a-x}$ as x approaches a can be investigated by setting $x = a - \delta$, where δ is a small positive number. This shows that $y \to \infty$ as $x \to a^-$ [ie x tends to a from below]. And $x = a + \delta$ shows that $y \to -\infty$ as $x \to a^+$.

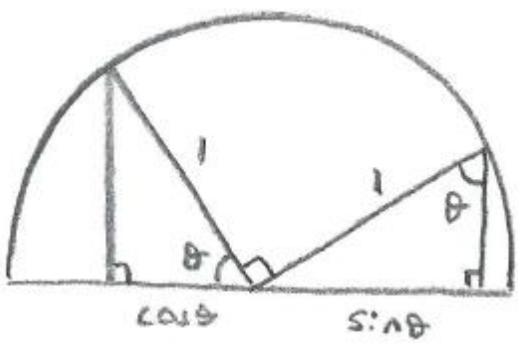
As $x \to \pm\infty, \mathrm{y} \to -1$



Then, with $-a < x < a$:



(ii) We can draw in radii as shown below, and discover that the area between the two (identical) triangles is a quarter of the circle.



So the area of A is $2\left(\frac{1}{2}\right)cos\theta sin\theta + \frac{1}{4}\pi(1)^2$

$= cos\theta sin\theta + \frac{\pi}{4}$ [or $\frac{1}{2}sin2\theta + \frac{\pi}{4}$]

(iii) $(sin\theta - cos\theta)^2 \geq 0 \Rightarrow sin^2\theta + cos^2\theta - 2sin\theta cos\theta \geq 0$

$\Rightarrow 1 \geq 2sin\theta cos\theta \Rightarrow sin\theta cos\theta \leq \frac{1}{2}$

(iv) $\frac{area\ of\ A}{area\ of\ B} = \frac{cos\theta sin\theta + \frac{\pi}{4}}{\frac{1}{2}\pi(1)^2 - \left(cos\theta sin\theta + \frac{\pi}{4}\right)} = \frac{cos\theta sin\theta + \frac{\pi}{4}}{\frac{\pi}{4} - cos\theta sin\theta}$

And $sin\theta cos\theta \leq \frac{1}{2} \Rightarrow \frac{\pi}{4} - cos\theta sin\theta \geq \frac{\pi}{4} - \frac{1}{2}$

$\Rightarrow \frac{cos\theta sin\theta + \frac{\pi}{4}}{\frac{\pi}{4} - cos\theta sin\theta} \leq \frac{\frac{1}{2} + \frac{\pi}{4}}{\frac{\pi}{4} - \frac{1}{2}} = \frac{2+\pi}{\pi-2}$ or $\frac{\pi+2}{\pi-2}$

And when $\theta = \frac{\pi}{4}$, $sin\theta cos\theta = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2}$,

So that when $\theta = \frac{\pi}{4}$, $\frac{area\ of\ A}{area\ of\ B} = \frac{\frac{1}{2} + \frac{\pi}{4}}{\frac{\pi}{4} - \frac{1}{2}} = \frac{2+\pi}{\pi-2}$

ie the ratio $\frac{area\ of\ A}{area\ of\ B}$ has $\frac{2+\pi}{\pi-2}$ as its largest value.