2013 MAT Paper - Q3 (3 pages; 26/11/20)

(i) $A(k) = \int_0^k x(x-k)(x-2)dx - \int_k^2 x(x-k)(x-2)dx$

(ii) $x(x-k)(x-2)$ is a cubic function in x which integrates to give a quartic in x, and when k is substituted for x in the limits, a quartic in k results (noting that the k in $x(x-k)(x-2)$ doesn't affect the cubic term in x, and will therefore only give rise to a cubic in k after integration). The sum of two such integrals will not involve any powers of k higher than k^4.

(iii) $f_k(1+t) + f_{2-k}(1-t)$

$= (1+t)(1+t-k)(1+t-2)$

$+(1-t)(1-t-[2-k])(1-t-2)$

$= (1+t)(1+t-k)(t-1) + (1-t)(-1-t+k)(-1-t)$

$= (1+t)(1+t-k)(t-1) + (1-t)(1+t-k)(1+t) = 0,$

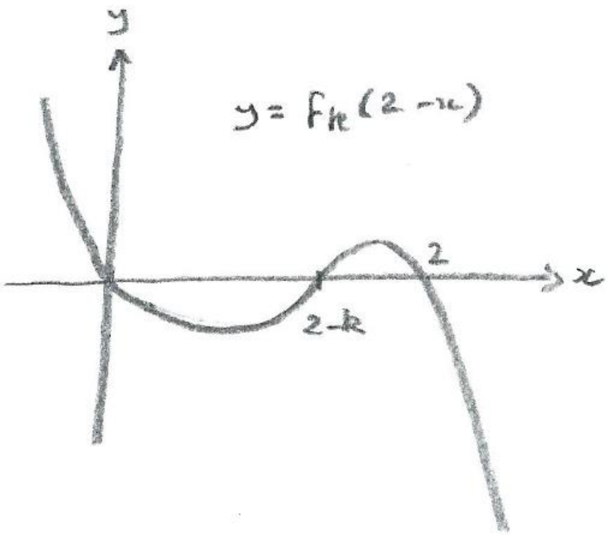
So that $f_k(1+t) = -f_{2-k}(1-t)$, as required.

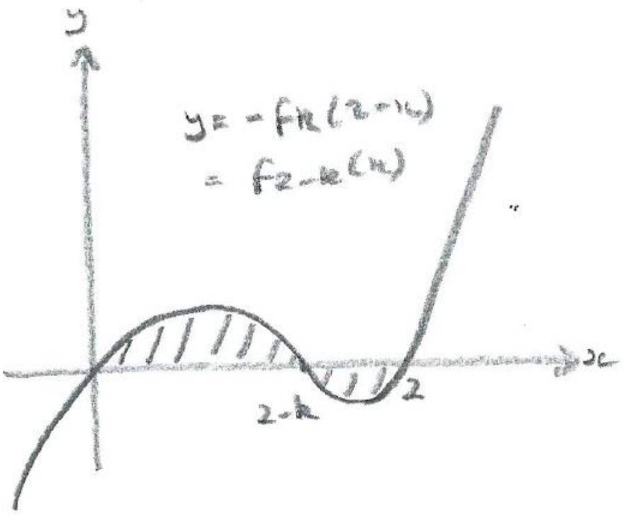
(iv) From (iii), $f_{2-k}(x) = -f_k(1+t)$, where $x = 1-t$, so that

$t = 1-x$ and $1+t = 2-x$

Thus $f_{2-k}(x) = -f_k(2-x)$

Replacing x with $2-x$ gives a reflection in $x = 1$, and so the required transformation is a reflection in $x = 1$, together with a reflection in the x-axis (see diagrams).





$A(2-k)$ is the area shaded in the diagram for $y = f_{2-k}(x)$, and this is the same as the shaded area for $y = f_k(x)$; ie $A(k)$.

[Line 3 of part (iv) of the official solution should read "x-axis " instead of "y-axis".]

(v) $A(k) = A(2-k) \Rightarrow y = A(x)$ is symmetric about $x = 1$

$(A(1-t) = A(2-[1-t]) = A(1+t))$

ie if $A(k)$ is translated by 1 to the left, an even function is produced

An even function, which is also a polynomial of degree 4 or less, will be of the form $y = ax^4 + bx^2 + c$,

And $y = A(x)$ will be obtained by translating this function by 1 to the right, giving $y = a(x-1)^4 + b(x-1)^2 + c$,

So that $A(k) = a(k-1)^4 + b(k-1)^2 + c$, as required.