2013 MAT Paper - Q2 (3 pages; 19/9/20)

Solution

(i) $f(t) - kf(1-t) = t \quad (1)$

Replacing t with $1-t$ gives $f(1-t) - kf(t) = 1-t \quad (2)$

Subst. from (2) into (1): $f(t) - k\{1 - t + kf(t)\} = t$

$\Rightarrow f(t)\{1-k^2\} = t + k(1-t) \quad (3)$

$\Rightarrow f(t) = \frac{t(1-k)+k}{1-k^2}$ (as $k^2 \neq 1$) [the eq'n of a straight line]

(ii)(a) If $f(t) - f(1-t) = t$, then from (3) in (i), with $k = 1$,

$0 = t + (1-t) = 1$

Hence $f(t) - f(1-t) = t$ is not possible.

(b) Suppose that $f(t) - f(1-t) = g(t) \quad (4)$

Then, replacing t with $1-t$ gives

$f(1-t) - f(t) = g(1-t) \quad (5)$

Adding (4) & (5): $0 = g(t) + g(1-t)$

Thus, $g(t)$ must satisfy the condition $g(1-t) = -g(t) \quad$ (A)

[Note that - in theory - there might be other conditions that $g(t)$ must satisfy. We have only shown that (A) is a necessary condition; not that it is a sufficient condition for (*). The question is slightly ambiguous: is 'condition' intended to relate to a single condition (one of possibly many). Or - if there were multiple conditions - does 'condition' mean 'the set of requirements'?

As a simpler example, if we were asked to give the condition that a number had to satisfy, in order for it to be a multiple of 9, we could say that the digits had to add up to a multiple of 3; but that wouldn't be in the spirit of the question.

At the end of the day, MAT questions are not concerned with the finer points of semantics.

For this question, it doesn't seem to be that straightforward to show that (A)$\Rightarrow$ ($*$); for example, by finding an expression for $f(t)$ in terms of $g(t)$, for which (*) held.]

(c) [The approaches mentioned in the official sol'n don't seem to be that obvious.]

If we suppose that $f(t)$ is a polynomial in t (bearing in mind that it is a straight line in (i)), then when we equate coefficients for (*), we will have a cubic (or lower order polynomial).

Suppose that $f(t) = At^3 + Bt^2 + Ct + D$,

so that $f(t) - f(1-t) = (At^3 + Bt^2 + Ct + D)$

$-[A(1-t)^3 + B(1-t)^2 + C(1-t) + D] = (2t-1)^3$ (B)

[At this stage, it is worth seeing if any useful observations can be made. Perhaps there is a simplified solution, with some of the coefficients equal to zero.]

As D cancels, it can take any value; such as 0.

[Expanding out everything and equating coefficients will be reliable, but quite time-consuming.]

Equating coefficients of t^3 in (B): $A + A = 8$, so that $A = 4$

Let $t = 1$ in (B), to give $A + B + C = 1$, and hence $B + C = -3$

[$t = 0$ gives the same result, unfortunately; as does $t = 2$]

Equating coefficients of t in (B): $C + 3A + 2B + C = 6$; once again giving the same result.

So it would appear that any cubic where $B + C = -3$ may well be a solution. Equating coefficients of t^2 will confirm this:

$B - 3A - B = -12$, which is consistent

So let $B = 0$ & $C = -3$, say, to give $f(t) = 4t^3 - 3t$

[and we can check that this satisfies (B)]