# 2013 MAT Paper - Multiple Choice (8 pages; 16/10/20)

## Q1/A

**Solution**

$x^2 + ax + a - 1 = 0$ has distinct real roots when

$a^2 - 4(a - 1) > 0$

ie when $(a - 2)^2 > 0$

ie for $a \neq 2$

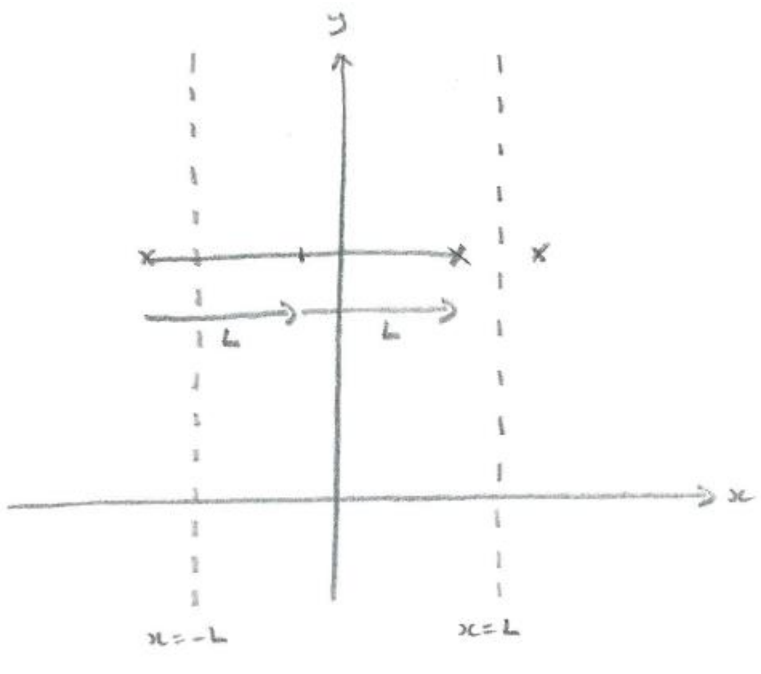
**So the answer is (a).**

## Q1/B

**Solution**

A reflection in the line $x = L$ is equivalent to a reflection in the line $x = 0$, followed by a translation of $\binom{2L}{0}$ (see the diagram below).

Thus $y = f(x) \rightarrow y = f(-x) \rightarrow y = f(-[x - 2L]) = f(2L - x)$



Similarly for a reflection in the line $y = M$, so that

$y = f(x) \to -y = f(x) \to -(y - 2M) = f(x)$ or $2M - y = f(x)$

So a reflection in $x = \pi$ gives

$y = \sin(2\pi - x) = \sin(-x) = -sinx$

Then a reflection in $y = 2$ gives

$4 - y = -sinx$ , or $y = sinx + 4$

**So the answer is (c).**

**Q1/C**

**Introduction**

This question really requires knowledge of the Chain Rule, for a full understanding.

As an example of a general choice of strategies, we could start with the information that we are given; ie $f'(x) = g(x + 1)$ etc, and aim to find $f''(2x)$ (this is what the official solution does), or we could start with $f''(2x)$ and try to use the information given.

The important point to note is that $f'(2x)$ means $\frac{d}{du}f(u)$, where $u = 2x$; so that the differentiation is with respect to $2x$ (rather than $x$).

**Solution**

$f''(2x) = f''(u)$, where $u = 2x$

and $f''(u) = \frac{d}{du}f'(u) = \frac{d}{du}g(w)$, where $w = u + 1$

Then $\frac{d}{du}g(w) = \frac{d}{dw}g(w).\frac{dw}{du} = \frac{d}{dw}g(w) = g'(w) = h(w - 1)$

So $f''(2x) = h(w-1) = h(u) = h(2x)$

**So the answer is (c).**

**Q1/D**

**Solution**

When $y = 0, x^4 - y^2 = 2y + 1 \Rightarrow x = \pm 1$

This eliminates (a).

When $x = 0, x^4 - y^2 = 2y + 1 \Rightarrow y^2 + 2y + 1 = 0$

$\Rightarrow (y+1)^2 = 0 \Rightarrow y = -1$

This eliminates (c) and (d).

**So the answer is (b).**

**Q1/E**

**Solution**

$f(x) = (2x-1)^4(1-x)^5$ is of degree 9,

so its 2nd derivative is of order 7

and $g(x) = (2x+1)^4(3x^2-2)^2$ is of degree 8, so its derivative is of order 7

Hence the given expression is of order 7, provided that the coefficients of $x^7$ don't cancel out.

For $f(x)$, the coefficient of $x^9$ is $-16$, and for $g(x)$, the coefficient of $x^8$ is $16(9) = 144$.

So the highest degree term in the given expression is

$-16(9)(8)x^7 + 144(8)x^7 = 0$

So the coefficients of $x^7$ do cancel,

**and hence the answer is (d).**

**Q1/F**

**Solution**

The 3 eq'ns can be rearranged to give:

$a = b^2$ , $c - 3 = b^3$ & $c + 5 = a^2$

The 2nd & 3rd eq'ns then give $b^3 + 3 = a^2 - 5$

If we try to obtain an eq'n in $a$ (as the multiple choice answers relate to $a$), we get

$a^3 = b^6 = (a^2 - 8)^2$ , which leads to

$a^4 - a^3 - 16a^2 + 64 = 0$ ,

and there are no obvious roots (ie trying $a = \pm 2$)

Instead, we can try to obtain an eq'n in $b$, and this gives

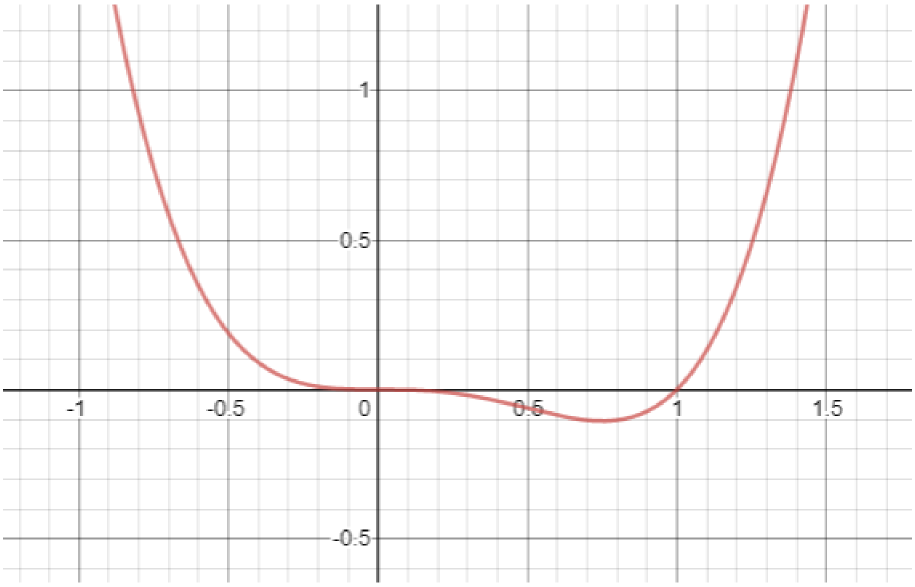
$b^3 + 3 = b^4 - 5$, so that $b^3(b - 1) = 8$

The graph of the quartic $y = x^3(x - 1)$ is shown below, and has a point of inflexion at $x = 0$

[Writing $f(x) = x^4 - x^3$; $f'(x) = 4x^3 - 3x^2$; $f''(x) = 12x^2 - 6x$

$f'''(x) = 24x - 6$

Thus $f''(0) = 0$ & $f'''(0) \neq 0$, which is a sufficient condition for a point of inflexion.]




So there is exactly one positive value of $b$ satisfying $b^3(b-1)=8$ and then one value of $a$ that satisfies $log_b a = 2$.

(Note that, $b > 0 \Rightarrow c - 3 = b^3 > 0$, so that $log_b(c-3)$ is defined.)

**Hence the answer is (a).**

**Q1/G**

**Solution**

$p_n(x) = nx - \frac{1}{2}n(n+1)$ and $p_{n-1}(x) = (n-1)x - \frac{1}{2}(n-1)n$

Let $p_n(x) = p_{n-1}(x)f(x) + R$

Now $p_{n-1}\left(\frac{1}{2}n\right) = 0$, so that $R = p_n\left(\frac{1}{2}n\right)$

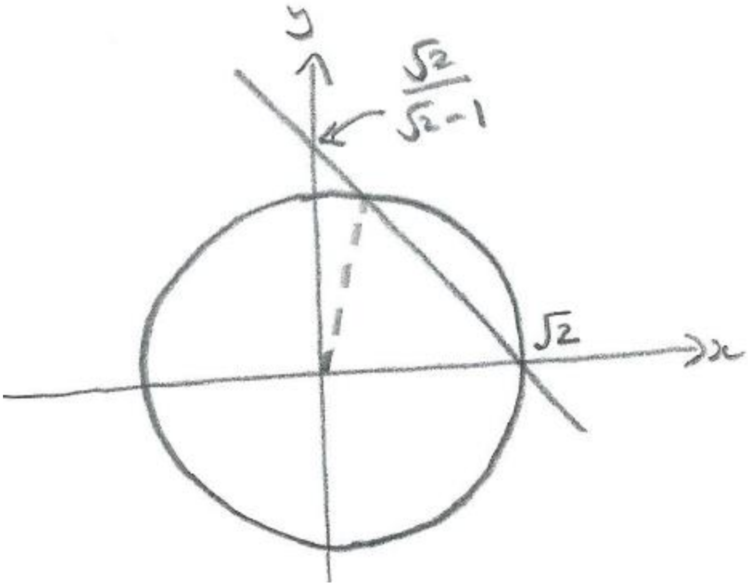
$= n\left(\frac{1}{2}n\right) - \frac{1}{2}n(n+1) = \frac{1}{2}n(-1) = -\frac{n}{2}$

**So the answer is (d).**

## Q1/H

### Solution

The required area is seen in the diagram below. (Note that the $y$-intercept is $\frac{\sqrt{2}}{\sqrt{2}-1} > \sqrt{2}$ )




Only one of the multiple choice options has a form consistent with the required area of a sector minus the area of a triangle.

**Thus the answer is (b).**

[The official sol'n just states that the line intersects the circle at (1,1), but this involves a certain amount of working to establish (with no guarantee of a convenient point of intersection).]

## Q1/I

### Solution

All the $F(r)$ are 1, except for the following, which are $-1$:

3

6, 7 [as $6 = 2(3)$ & $7 = 2(3) + 1$]

12, 13; 14, 15 [as $12 = 2(6)$ & $13 = 2(6) + 1$ etc]

24, 25; 26, 27; 28, 29; 30, 31 [8 numbers]

48, 49; 50 ... [16 numbers]

96, 97; 98, 99; 100 [5 numbers]

So there are $1 + 2 + 4 + 8 + 16 + 5 = 36$ with value $-1$,

and hence 64 with value 1

Hence $F(1) + F(2) + \cdots + F(100) = 64 - 36 = 28$

**So the answer is (b).**

**Q1/J**

**Solution**

[Answer = (b); not (d) as stated in the official solution (though the solution itself is correct).]

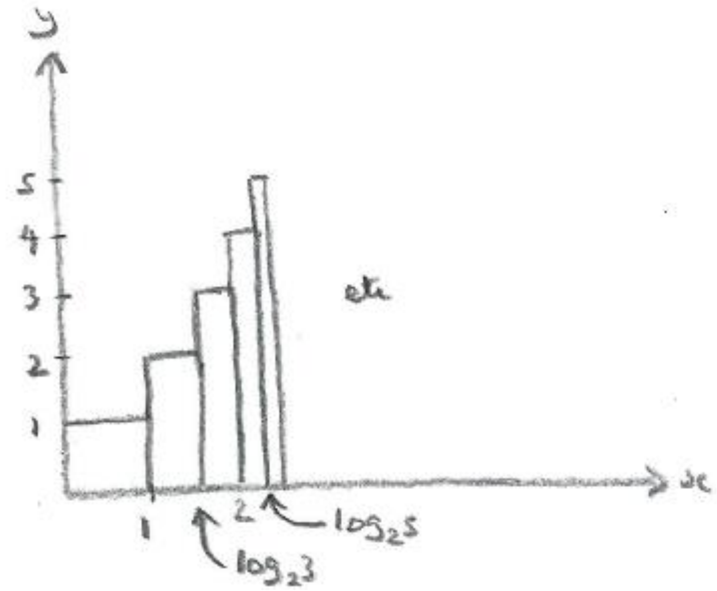
The 'floor' function $[x]$ is sometimes written as $\lfloor x \rfloor$, with the 'ceiling' function (the smallest integer greater than or equal to $x$), being written as $\lceil x \rceil$ (though these latter symbols do look like printing errors).

Note that $2^0 = 1, 2^1 = 2, 2^{log_2 3} = 3, \ldots, 2^{log_2 r} = r$

So, for $0 \leq x < 1$, $[2^x] = 1$

For $1 \leq x < log_2 3$, $[2^x] = 2$

For $log_2 3 \leq x < log_2 4 = 2$, $[2^x] = 3$ etc




Summing the rectangles represented by the integral gives:

$\int_0^3 [2^x] dx = 1(log_2 2) + 2(log_2 3 - log_2 2) + 3(log_2 4 - log_2 3) +$...

$+7(log_2 8 - log_2 7)$ )

$= 7log_2 8 - log_2 7 - log_2 6 - \cdots - log_2 2,$ so that

$\int_0^n [2^x] dx = (2^n - 1)log_2(2^n) - log_2(2^n - 1) - log_2(2^n - 2) - \cdots - log_2 2$

$= (2^n - 1)n - log_2((2^n - 1)!)$

[Referring to the multiple choice options:]

$= n2^n - log_2 2^n - log_2((2^n - 1)!)$

$= n2^n - log_2((2^n)!)$

**So the answer is (b).**