2010 MAT Paper - Multiple Choice (7 pages; 28/18/20)

Q1/A

Solution

At the point of intersection, $(x-1)^2 = kx$,

or $x^2 - (k+2)x + 1 = 0$

A solution exists when $(k+2)^2 - 4 \geq 0$

$\Rightarrow k^2 + 4k \geq 0$

$\Rightarrow k(k+4) \geq 0$

$\Rightarrow k \geq 0$ or $k \leq -4$ (from the graph of $y = k(k+4)$, for example).

So the answer is (c).

Q1/B

Solution

$1 + 1 + 2 + \frac{1}{2} + 4 + \frac{1}{4} + 8 + \frac{1}{8} + 16 + \frac{1}{16} + \cdots 2^{n-1} + \frac{1}{2^{n-1}}$

$= (1 + 2 + 4 + 8 + 16 + \cdots + 2^{n-1})$

$+(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots + \frac{1}{2^{n-1}})$

$= \frac{2^n - 1}{2-1} + \frac{1-(\frac{1}{2})^n}{1-\frac{1}{2}}$

$= (2^n - 1) + 2(1 - 2^{-n})$

$= 2^n + 1 - 2^{1-n}$

So the answer is (a).

Q1/C

Solution

Dividing by cos^2x gives

$tan^2x + 3tanx + 2 = 0$, provided that $cosx \neq 0$

If $cosx = 0$, then the original eq'n gives $sin^2x = 0$, which has no sol'ns when $cosx = 0$.

So $(tanx + 1)(tanx + 2) = 0$

$\Rightarrow tanx = -1\ or - 2$

giving 4 sol'ns in the range $0 \leq x < 2\pi$

So the answer is (d).

Q1/D

Solution

Maxima occur at $\sqrt{x} = (2k + 1)\left(\frac{\pi}{2}\right)$, so that $x = (2k + 1)^2\left(\frac{\pi^2}{4}\right)$; ie the gap between x-values increases (and the value of the maximum is always 1),

so the answer must be (b).

Q1/E

Solution

Let $log_2 3 = a$ etc

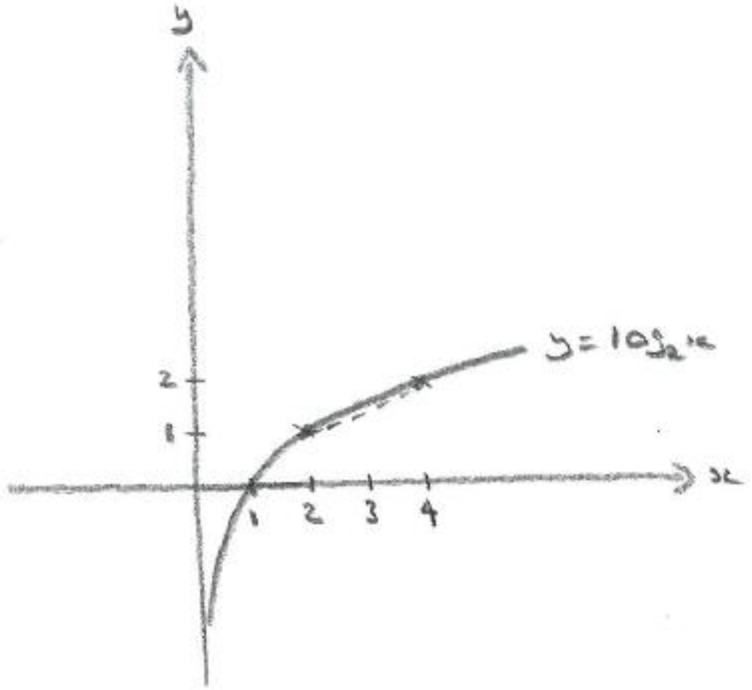
There doesn't seem to be any general procedure for deciding on the relative sizes of $log_p x$ & $log_q y$. But there are various ad-hoc results that we can use. For example, it is often possible to establish whether $log_p x$ is greater or less than a particular value, as shown below.

In this case, we note first of all that

$1 < a < 2, 1 < b < 2, \ 0 < c < 1$ & $1 < d < 2$, so that (c) can be eliminated.

Also $b = log_4(2^3) = 3log_4 2 = 3\left(\frac{1}{2}\right) = 1.5$

Now, from the diagram below we see that $a = log_2 3 > 1.5$



So that (b) can be eliminated.

Then $log_5 10 = log_5(5 \times 2) = log_5 5 + log_5 2 = 1 + log_5 2$

$< 1 + log_5\sqrt{5} = 1.5$, so that $d < 1.5 < a$

and **the answer is therefore (a).**

Q1/F

Solution

The Trapezium rule estimate will equal the actual integral when the interval borders include $x = \frac{1}{3}, \frac{1}{2}$ & $\frac{3}{4}$. This means that n has to be a multiple of 12.

So the answer is (d).

Q1/G

Solution

$f(2) = 2f(1) = 2$

$f(3) = 4f(1) = 4$

$f(4) = 2f(2) = 4$

$f(5) = 4f(2) = 8$

$f(6) = 2f(3) = 8$

$f(7) = 4f(3) = 16$ *

$f(8) = 2f(4) = 8$

$f(9) = 4f(4) = 16$ *

$f(10) = 2f(5) = 16$ *

$f(11) = 4f(5) = 32$

[and subsequent $f(2n + 1)$ will also exceed 16, as $f(n)$ will exceed 4 when n exceeds 4]

$f(12) = 2f(6) = 16$ *

$f(14) = 2f(7) = 32$

$f(16) = 2f(8) = 16$ *

$f(18) = 2f(9) = 32$

[and subsequent $f(2n)$ will also exceed 16, as $f(n)$ will exceed 8 when n exceeds 8]

Thus there are 5 values of n that satisfy $f(n) = 16$.

So the answer is (c).

Q1/H

Solution

(a): All cubic eq'ns with real coefficients have at least one root, so that (a) can be eliminated.

(b): Considering the shape of the quartic on the LHS (starting in the 2nd quadrant, and finishing in the 1st quadrant), there will not be a sol'n of the eq'n if k is sufficiently large and negative.

Thus (b) can be eliminated.

(d): Consider $(x-1)(x-2) = k$

$\Rightarrow x^2 - 3x + 2 - k = 0$

$\Rightarrow (x - \frac{3}{2})^2 - \frac{9}{4} - k = 0$

Then let $k = -\frac{9}{4}$, so that $x = \frac{3}{2}$ is a repeated root.

Thus (d) can be eliminated.

So the answer must be (c).

[To show that (c) is true: If n is odd, then the polynomial function on the LHS starts in the 3rd quadrant, and finishes in the 1st quadrant, so that it must cross $y = k$. If n is even, then it starts in the 2nd quadrant, and finishes in the 1st quadrant (and crosses the x-axis), so that it must cross $y = k$ if $k \geq 0$.]

Q1/I

Introduction

This question is based on the 'Fundamental Theorem of Calculus':

If $F(x) = \int_a^x f(t)dt$ [with different notation from the question], then $F'(x) = f(x)$.

It amounts to saying that integration is the opposite of differentiation.

Solution

For this question, $\frac{dI}{dA} = 0 \Rightarrow 4 - 2^{a^2} = 0 \Rightarrow a^2 = 2 \Rightarrow a = \sqrt{2}$ (as $a > 0$).

So the answer is (b).

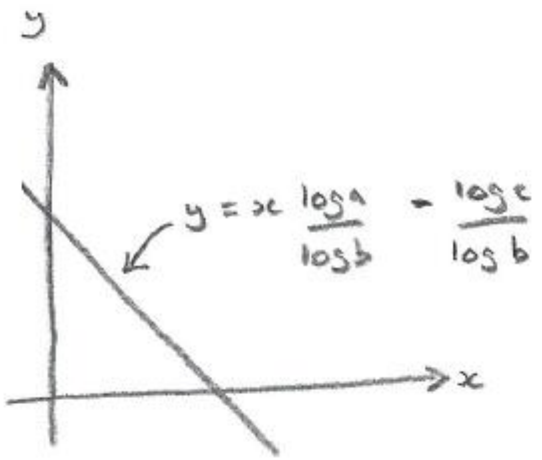
Q1/J

Solution

$a^x > cb^y \Rightarrow xloga > logc + ylogb$,

or $y < x\frac{loga}{logb} - \frac{logc}{logb}$

(The log can be to any base, but 'log' usually means to the base 10.)



Referring to the diagram, we require the gradient of the line

$y = x\frac{loga}{logb} - \frac{logc}{logb}$ to be negative, in order for there to be a finite number of integer pairs (x, y) satisfying the inequality (with x & y being positive).

(a) allows $a < 1$ & $b < 1$, when $loga < 0$ & $logb < 0$, so that

$\frac{loga}{logb} > 0$, and (a) can therefore be eliminated.

(b) also allows $a < 1$ & $b < 1$, and can also be eliminated, as can (c).

For (d), $loga < 0$ & $logb > 0$, so that $\frac{loga}{logb} < 0$, as required.

So the answer is (d).