2009 MAT - Multiple Choice (6 pages; 28/8/20)

Q1/A

Solution

$I(a) = \int_0^1 (x^2 - a)^2 \, dx = [\frac{1}{5}x^5 - 2a.\frac{1}{3}x^3 + a^2 x]_0^1$

$= \frac{1}{5} - \frac{2}{3}a + a^2$

$= \frac{1}{5} + (a - \frac{1}{3})^2 - \frac{1}{9}$

Hence the smallest value of $I(a)$ is $\frac{1}{5} - \frac{1}{9} = \frac{4}{45}$

So the answer is (b).

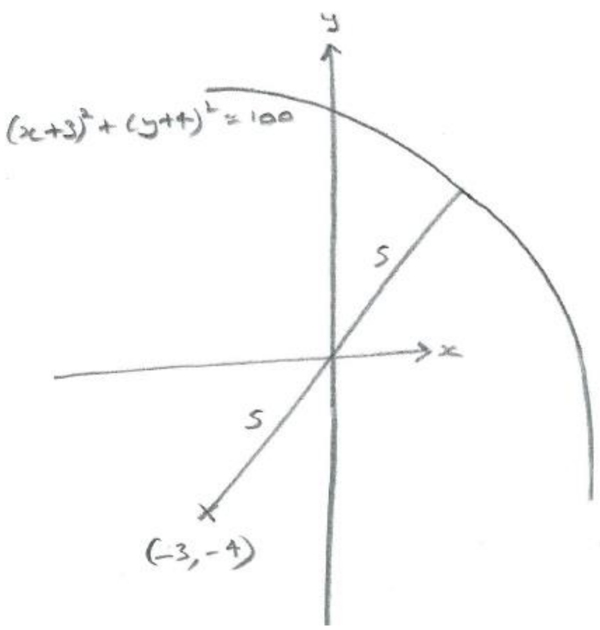
Q1/B

Solution

$x^2 + y^2 + 6x + 8y = 75 \Rightarrow (x+3)^2 - 9 + (y+4)^2 - 16 = 75$

$\Rightarrow (x+3)^2 + (y+4)^2 = 100$

ie a circle centre $(-3, -4)$ with radius 10



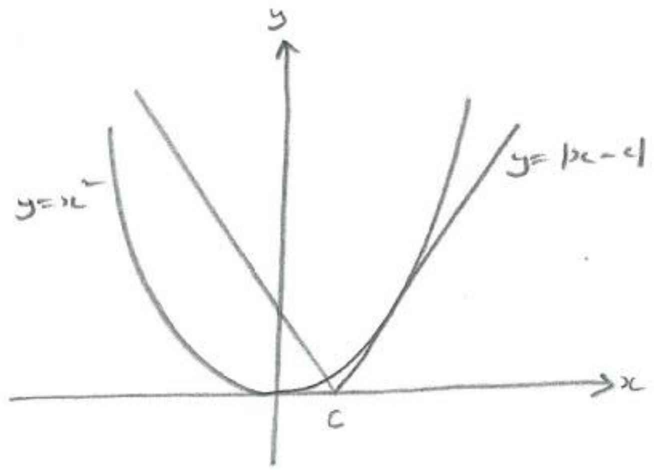
Referring to the diagram, the required point will be on the radius passing through the Origin. Hence the distance is 5.

So the answer is (c).

Q1/C

Solution

$x^4 = (x-c)^2 \Rightarrow x^2 = |x-c|$



The diagram show the critical point at which the number of roots changes from 4 to 2 (for larger values of c). By symmetry, if the critical value of c is c_1, then $-c_1$ will also be a critical value (with the number of roots being 2 for $c < -c_1$. As there is only one answer of the form $-c_1 \leq c \leq c_1$,

the answer must be (b).

[As a check, the gradients of $y = x^2$ & $y = x - c$ are equal when

$2x = 1$; ie $x = \frac{1}{2}$, so that the line $y = x - c$ has to pass through

the point $(\frac{1}{2}, \left(\frac{1}{2}\right)^2)$, and hence $\frac{1}{4} = \frac{1}{2} - c$, giving $c = \frac{1}{4}$]

Q1/D

Solution

[Because of the presence of $(-1)^{n+1}$, it is worth considering separately even and odd n.]

With even n, the LHS becomes $1-2+3-4+\cdots-2m$, writing $n=2m$.

By grouping the terms as $(1-2)+(3-4)+\cdots-2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1-2+3-4+\cdots+(2m+1)$, writing $n=2m+1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$=m(-1)+(2m+1)=m+1$

So we want $m+1\geq 100$, and hence

$n=2m+1\geq 2(99)+1=199$

So the answer is (c).

Q1/E

Solution

$2^{sin^2x}+2^{cos^2x}=2 \Rightarrow 2^{sin^2x}+2^{1-sin^2x}=2$

Let $y=2^{sin^2x}$, so that $y+\frac{2}{y}=2$

and $y^2-2y+2=0$

As the discriminant of these quadratic is negative, there are no sol'ns.

So the answer is (a).

Q1/F

Solution

We can investigate the intersection of

$y = 3x^4 - 16x^3 + 18x^2$ and $y = -k$

To simplify matters, we can make the substitution $c = -k$

A quick sketch reveals that the answer has to take the form

$0 < c < a$, and so $-a < k < 0$; ie **the answer is (d)**, without further calculations (though it could be found directly by determining stationary points).

Q1/G

Solution

$sinx = siny \Rightarrow y = x \ \pm 2k\pi \ or \ y = \pi - x \ \pm 2k\pi$

So the answer is (c).

Q1/H

Solution

Trapezium rule estimate for $\int_0^1 2^x \, dx$ is

$\frac{1}{2}\left(\frac{1}{N}\right)\{2^0 + 2^1 + 2\left(2^{\frac{1}{N}} + 2^{\frac{2}{N}} + \cdots + 2^{\frac{N-1}{N}}\right)\}$

$= \frac{1}{2N}\{1 + 2(1 + 2^{\frac{1}{N}} + \cdots + 2^{\frac{N-1}{N}}\}$

$= \frac{1}{2N}\{1 + \frac{2\left(\left(2^{\frac{1}{N}}\right)^{N} - 1\right)}{2^{\frac{1}{N}} - 1}\}$

$= \frac{1}{2N}\{1 + \frac{2}{2^{\frac{1}{N}} - 1}\}$

So the answer is (b).

Q1/I

Solution

$x^2 - 1$ will be a factor when $x - 1$ & $x + 1$ are both factors

Writing $f(x) = n^2x^{2n+3} - 25nx^{n+1} + 150x^7$,

the Factor theorem gives

$f(1) = n^2 - 25n + 150 = 0$

and $f(-1) = -n^2 - 25n - 150 = 0$ if n is odd

or $-n^2 + 25n - 150 = 0$ if n is even,

with no sol'n if n is an integer [it's possible that the question-setter forgot to say that n has to be an integer]

So, for odd n, $n^2 - 25n + 150 = 0$ & $n^2 + 25n + 150 = 0$,

which isn't possible (as $n = 0$ isn't possible, because $x^2 - 1$ isn't a factor of $150x^7$).

For even n, $n^2 - 25n + 150 = 0$ or $(n - 15)(n - 10) = 0$,

so that $n = 15$ (reject, as not even) or 10

So the answer is (b).

Q1/J

Solution

The presence of $8y^3$ suggests that $(x + 2y)^3$ might possibly expand to give the LHS - which it does.

This then gives $x + 2y = 2^{10}$, and we can simplify matters by writing $x = 2u$ (since x has to be even), to give $u + y = 2^9$.

Then y can take the values $1, 2, \ldots, 2^9 - 1$ (with $x = 2^{10} - 2y$), so that there are $2^9 - 1$ such pairs.

So the answer is (c).