2008 MAT - Q5 (4 pages; 5/10/22)

Solution

(i) The lockers that are closed after the 3rd student are those that:

are even [and were closed after the 2nd student], and not a multiple of 3; or

are odd [and were open after the 2nd student], and are a multiple of 3

Thus: (2, 3, 4), (8, 9, 10), (14, 15, 16), ... , (998, 999, 1000)

ie 3 for every alternate multiple of 3

As 999 is the 333rd multiple of 3, and the $\frac{332}{2} + 1 = 167$th in the sequence of multiples of 3 that are included,

the answer is $3 \times 167 = 501$

(ii) The lockers that are closed after the 4th student are those that:

(a) are closed after the 3rd student, and are not multiples of 4; or

(b) are open after the 3rd student, and are multiples of 4

Thus, for (a): (2, 3), (9, 10), (14, 15), ... , (998, 999),

giving 2 for every alternate multiple of 3

As 999 is the 333rd multiple of 3, and the $\frac{332}{2} + 1 = 167$th in the sequence of multiples of 3 that are included,

the total for (a) is $2 \times 167 = 334$

The lockers that are open after the 3rd student are:

1, (5, 6, 7), (11, 12, 13), (17, 18, 19), (23, 24, 25), ..., (995, 996, 997),

and of these the multiples of 4 are:

12, 24, ..., 996 ($= 83 \times 12$) ; ie multiples of 12

Thus the total for (b) is 83, and so the answer is $334 + 83 = 417$

Alternative method

The table below shows the outcome for the different lockers. The locker will be closed if there are 1 or 3 Ys. The cycle repeats itself after 12.

	multiple of 2	multiple of 3	multiple of 4	outcome
1	X	X	X	Open
2	Y	X	X	Closed
3	X	Y	X	Closed
4	Y	X	Y	Open
5	X	X	X	Open
6	Y	Y	X	Open
7	X	X	X	Open
8	Y	X	Y	Open
9	X	Y	X	Closed
10	Y	X	X	Closed
11	X	X	X	Open
12	Y	Y	Y	Closed

Thus 5 of the lockers from 1-12 will be closed after the 4th student.

As $83 \times 12 = 996$, the answer is:

$(83 \times 5) + 2$ (there being 4 lockers in the 84th cycle of 12, of which the 2nd & 3rd will be Closed)

$= 417$

(iii) Every factor of 100 results in a change of state of the 100th locker.

The factors are: 2, 4, 5, 10, 20, 25, 50 & 100

As there are an even number of these, the 100th locker will in the same state as after the 1st student; ie Open.

(iv) Every factor of 1000, up to 100, results in a change of state of the 1000th locker.

The prime factorisation of 1000 is $2^3 \times 5^3$, which can help in producing a systematic list of the required factors:

$5, 5^2, 2, 2 \times 5, 2 \times 5^2, 2^2, 2^2 \times 5, 2^2 \times 5^2, 2^3, 2^3 \times 5$

Or $5, 25, 2, 10, 50, 4, 20, 100, 8, 40$

As there are an even number of these, the 1000th locker will in the same state as after the 1st student; ie Open.

[The examiners have refrained from asking for the state of the 1000th locker after the 1000th student has walked past. (Perhaps they intended to originally).

Let $f(n)$ be the number of factors (other than 1) of the number n.

Then it can be seen that, if m & n have no common factors,
$f(mn) = (f(m) + 1)(f(n) + 1) - 1$

or $f(m)f(n) + f(m) + f(n)$

For example, with $m = 25$ & $n = 4$, the factors of 100 are formed by combining one factor from the set $\{1, 5, 25\}$ and one from the set $\{1, 2, 4\}$, but then discarding the number 1; ie there are

$3 \times 3 - 1 = 8$ factors, as found in (iii).

Then $f(1000) = f(2^3 \times 5^3) = (f(2^3) + 1)(f(5^3) + 1) - 1$

$= (3 + 1)(3 + 1) - 1 = 15$

As there are an odd number of these factors, the 1000th locker will be in the opposite state as after the 1st student; ie Closed.]