2008 MAT - Q3 (2 pages; 27/8/20)

Solution

(i) (A) $y = -f(x)$ (B) $y = f(-x)$ (C) $y = f(x-1)$

(ii) [$y = 2^{-x^2}$ will have similarities to $y = 2^{-|x|}$, which can be obtained from $y = 2^x$ & $y = 2^{-x}$;

compare $y = 2^{-x^2}$ with the standardised normal pdf:

$y = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$]

Although outside the MAT syllabus, we could find the gradient of $y = 2^{-x^2}$ at $x = 0$ as follows:

$\frac{d}{dx}(a^x) = (lna)a^x$

Setting $y = -x^2$,

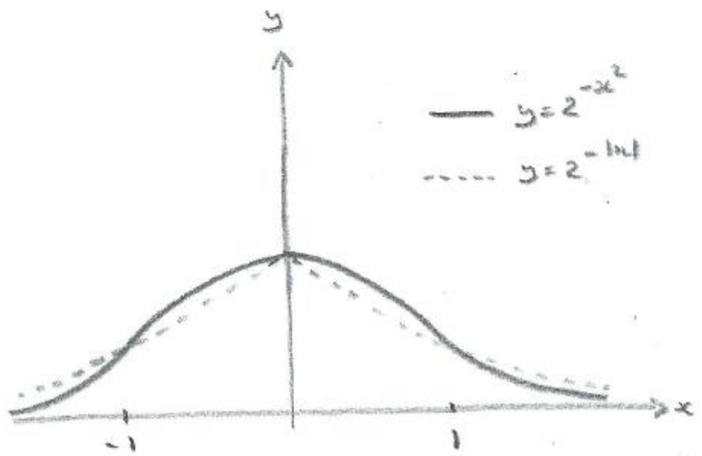
$\frac{d}{dx}\left(2^{-x^2}\right) = \frac{d}{dy}(2^y)\frac{dy}{dx} = (ln2)(2^y)(-2x) = -2(ln2)x\left(2^{-x^2}\right)$

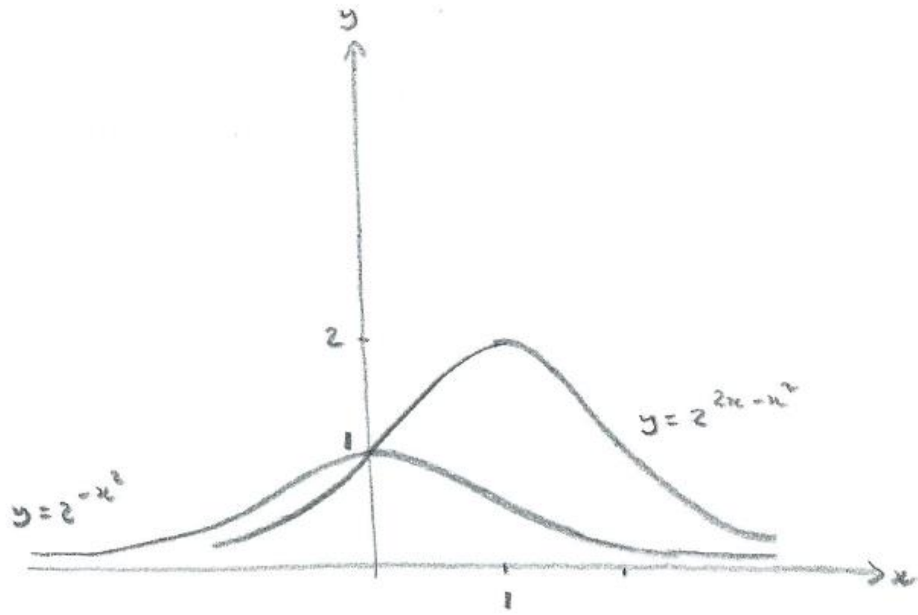
So gradient of $y = 2^{-x^2}$ at $x = 0$ is 0.

Also, the gradient is a continuous function.

Comparing $y = 2^{-x^2}$ with $y = 2^{-|x|}$:

$2^{-\frac{1}{4}} > 2^{-\frac{1}{2}}$ & $2^{-4} < 2^{-2}$





Then $y = 2^{2x-x^2} = 2^{1-(x-1)^2} = (2)2^{-(x-1)^2}$, which is obtained from $y = 2^{-x^2}$ by a translation of $\binom{1}{0}$, followed by a stretch in the y-direction of scale factor 2 (or the other way round).

[Note that part (iii) mentions $2^{-(x-c)^2}$, suggesting the above approach.]

(iii) $y = 2^{-(x-c)^2}$ is obtained from $y = 2^{-x^2}$ by a translation of $\binom{c}{0}$, and has its maximum at $x = c$. The area under the curve between 0 and 1 (equal to the integral) will be maximised when $c = 1/2$, with the maximum value of the function $y = 2^{-(x-c)^2}$ being in the middle - due to the symmetry of the curve.