# 2008 MAT - Multiple Choice (6 pages; 27/8/20)

## Q1/A

### Introduction

In general, cubics can have 0, 1 or 2 stationary points. See separate note on the Pure page.

### Solution

Let $f(x) = 2x^3 - 6x^2 + 5x - 7$

Then $f'(x) = 6x^2 - 12x + 5$

Stationary points occur when $f'(x) = 0$

The discriminant of the quadratic is $144 - 120 > 0$,

and so there are 2 stationary points.

**So the answer is (c).**

## Q1/B

### Solution

Write $L = log_{10}\pi$

[It often helps to have a rough idea of the sizes of the multiple choice options.]

$L \approx \frac{1}{2}$, so that $(b) = \sqrt{2L} \approx 1, (c) = \left(\frac{1}{L}\right)^3 = 8, (d) = \frac{1}{\frac{1}{2}L} \approx 4$

so that we can be fairly sure that the answer is (a), and might just like to check that (a) < (b):

result to prove: $L < \sqrt{2L}$

As $L < 1$ (as $\pi < 10$), $\sqrt{2L} = \sqrt{2}\sqrt{L} > \sqrt{2}\,L > L$, as required.

**So the answer is (a).**

## Q1/C

### Solution

### Method 1

A shortcut can be taken here if you have studied Matrices.

As $\begin{vmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{vmatrix} = cos^2\theta + sin^2\theta = 1 \neq 0$, a (unique) solution to the simultaneous equations exists for all $\theta$.

### Method 2

Alternatively, the equations can be interpreted as the matrix equation

$$\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix},$$

representing an anti-clockwise rotation of $\theta$ rad. As there will always be a point which, when rotated through angle $\theta$, gives the point $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$, there will always be a solution to the equation.

### Method 3

Writing $c = cos\theta$ & $b = sin\theta$ [$'s'$ looks too much like $'5'$ when written by hand],

$cx - by = 2$ & $bx + cy = 1$ (A)

$\Rightarrow c^2x - bcy = 2c$ & $b^2x + bcy = b$

Adding these gives $c^2x + b^2x = 2c + b$,

so that $x = \frac{2c+b}{c^2+b^2} = 2c + b$

Also (from (A)), $bcx - b^2y = 2b$ & $bcx + c^2y = c$,

and subtracting the 1st eq'n from the 2nd gives

$c^2y + b^2y = c - 2b$, so that $y = \frac{c-2b}{c^2+b^2} = c - 2b$

Thus solutions exist for any $\theta$.

**So the answer is (a).**

**Q1/D**

**Solution**

Let $f(x) = 1 + 3x + 5x^2 + 7x^3 + \cdots + 99x^{49}$

By the Remainder theorem, the remainder when $f(x)$ is divided by $x - 1$ is $f(1) = 1 + 3 + 5 + 7 + \cdots + 99$

$= (1 + 2 + 3 + \cdots + 100) - (2 + 4 + 6 + \cdots + 100)$

$= \frac{1}{2}(100)(101) - 2(\frac{1}{2})(50)(51)$

$= 50(101 - 51)$

$= 2500$

**So the answer is (b).**

**Q1/E**

**Solution**

The highest power will be the same as the highest power in

$\{(x^{18} + x^{32})^5 + (x^{25} + x^{28})^6\}^3$

or in $\{x^{160} + x^{168}\}^3$;

giving $x^{504}$

**So the answer is (d).**

## Q1/F

### Solution

If there is an overestimate for $\int_0^1 f(x)dx$, then $f(x)$ is predominantly convex (shaped as $y = e^x$ [aid to memory: conv$e^x$], rather than the concave $y = lnx$). Of the 4 options, only $1 - f(x)$ becomes predominantly concave, being a reflection in the line $y = \frac{1}{2}$ [In general, replacing $x$ or $y$ with $2L - x$ or $2L - y$ (respectively) produces a reflection in the line $x = L$ or $y = L$.]

**So the answer is (d).**

## Q1/G

### Introduction

The 1st step with this type of question is to look for simple similarities and differences between the graphs which can easily be investigated.

### Solution

When $x = 0$, (a) and (b) give $y > 0$, whilst (c) and (d) give $y < 0$. As $x = 0 \Rightarrow y = -\frac{1}{5}$ for the function in question, we can therefore eliminate (a) and (b). Alternatively, as $x \to \infty$, $\frac{1}{4x-x^2-5} \to 0^-$.

The main distinguishing feature between (c) and (d) is then the sign of $x$ at the minimum point.

$4x - x^2 - 5 = -(x-2)^2 - 1$, so that $\frac{1}{4x-x^2-5} = -\frac{1}{(x-2)^2+1}$, and in order to obtain a minimum we want $\frac{1}{(x-2)^2+1}$ to be as large as possible; ie $(x-2)^2 + 1$ needs to be as small as possible, and hence $x = 2$;

**so that the answer is (c).**

## Q1/H

**Solution**

Let $y = 3^x$.

Then $9^x - 3^{x+1} = k \Rightarrow y^2 - 3y - k = 0$

For there to be one or more real sol'ns, the discriminant must be non-negative;

ie $9 + 4k \geq 0$, so that $k \geq -\frac{9}{4}$

Then $y = \frac{3 \pm \sqrt{9+4k}}{2}$ , and at least one of these values will be positive, so that a sol'n exists for $x$.

**So the answer is (a).**

## Q1/I

**Solution**

[The official solution is based on spotting the 'lateral thinking' idea that, by symmetry, there must be 20 of each of the 10 digits

0, 1, 2, ..., 9 amongst the numbers 00, 01, ..., 99 (there being 200 digits in total). Alternative method:]

$S(1) + \cdots + S(9) = \sum_{i=1}^{9} i = \frac{1}{2}(9)(10) = 45$

$S(10) + \cdots + S(19) = (10 \times 1) + \sum_{i=1}^{9} i = 10 + 45$

[not writing 55 at this stage, so as not to lose the 45, which is clearly going to be cropping up again]

$S(20) + \cdots + S(29) = (10 \times 2) + \sum_{i=1}^{9} i = 20 + 45$ ,

and so on, giving a grand total of

$(10 + 20 + \cdots + 90) + (10 \times 45) = 10(45) + 450 = 900$

**So the answer is (c).**

## Q1/J

### Introduction

This equation can be interpreted as the intersection of the functions $y = (3 + cosx)^2$ and $y = 4 - 2sin^8x$.

In order for the functions to intersect, their ranges must overlap.

Given the relatively complicated nature of these functions, the simplest outcome for this question would be that either the ranges don't overlap at all, or they overlap at one value.

### Solution

Note that $(3 + cosx)^2 \geq 4$,

whilst $4 - 2sin^8x \leq 4$

So a sol'n only exists when $cosx = -1$ and $sinx = 0$;

ie at $x = \pi$ (in the range $0 \leq x < 2\pi$)

**So the answer is (b).**