2007 MAT – Q6 (4 pages;17/11/23)

Solution

(i) Let $A = 1$ denote A always tells the truth, $A = \frac{1}{2}$ denote A tells truth or lies at random, and $A = 0$ denote A always lies.

From A's statement,

$A = 1 \Rightarrow A = 0$; ie a contradiction

$A = \frac{1}{2}$ & currently telling the truth $\Rightarrow A = 0$; ie a contradiction

$A = \frac{1}{2}$ & currently lying $\Rightarrow A = 1 \; or \; \frac{1}{2}$; ie consistent with $A = \frac{1}{2}$

$A = 0 \Rightarrow A = 1 \; or \; \frac{1}{2}$; ie a contradiction

So $A = \frac{1}{2}$

Then B's statement implies that $B = 0$ (B cannot $= \frac{1}{2}$, as $A = \frac{1}{2}$).

Hence $G = 1$.

So Alf tells the truth or lies at random, Beth always lies, and Gemma always tells the truth.

(ii) Define the following states [to improve the notation!]:

$X = 1$: X always tells the truth

$X = 2$: X always lies

$X = 3$: X tells the truth or lies at random, and is currently telling the truth

$X = 4$: X tells the truth or lies at random, and is currently lying

Suppose that $G = 1$. Then, from G's statement, $B = 1$. But it isn't possible for both B & G to equal 1. So $G \neq 1$.

Suppose instead that $G = 2$. Then, from G's statement, $B \neq 1$. And

it isn't possible for both B & G to equal 2. Also, from B's statement, B cannot equal 4, but could equal 3.

So one solution is that $G = 2, B = 3$, and therefore $A = 1$.

Suppose instead that $G = 3$. Then, from G's statement, $B = 1$. But this is contradicted by B's statement. So $G \neq 3$.

Suppose instead that $G = 4$. Then, from G's statement, $B \neq 1$. And

it isn't possible for B to equal 3 or 4 (ie tell the truth or lie at random)(as $G = 4$). But $B = 2$ is inconsistent with B's statement. So $G \neq 4$.

Thus the only solution is that $A = 1, B = 3$ and $G = 2$;

ie Alf always tells the truth, Gemma always lies, and Beth tells the truth or lies at random

(iii) **Suppose that $\boldsymbol{A = 1}$.**

Then Alf's statement implies that $B = 3 \; or \; 4$.

Gemma's statement implies that $G = 2 \; or \; 4$, so that either

$B = 3$ & $G = 2$ (as if one person is 3 or 4, then no one else can be 3 or 4), or $B = 4$ & $G = 2$

Thus $A = 1, B = 3 \; or \; 4, G = 2$

Beth's statement is then consistent with this, as she could be telling the truth or lying.

So $\boldsymbol{A = 1, B = 3 \; or \; 4, G = 2}$ is a possibility (*)

Suppose instead that $\boldsymbol{A = 2}$.

Then Alf's statement implies that $B = 1$, and so $G = 3\ or\ 4$.

And Gemma's statement implies that $G = 3$.

If $B = 1$, then Beth's statement means that we can deduce who is telling the truth. But this is contradicted by the fact that it is possible that either $A = 1\ \big(from\ (*)\big)\ or\ B = 1$.

So $\boldsymbol{A \neq 2}$.

Suppose instead that $\boldsymbol{A = 3}$.

But this is contradicted by Alf's statement.

So $\boldsymbol{A \neq 3}$.

Suppose instead that $\boldsymbol{A = 4}$.

This is then consistent with Alf's statement.

Then either $B = 1$ & $G = 2$ (A) or $B = 2$ & $G = 1$ (B)

If (A) is true, then Gemma's statement implies that $A \neq 2$ (which is consistent with $A = 4$). But, as before, Beth's statement means that we can deduce who is telling the truth. But this is contradicted by the fact that it is possible that either

$A = 1\ \big(from\ (*)\big)\ or\ B = 1$.

If instead (B) is true, then Gemma's statement implies that $A = 2$, which contradicts $A = 4$.

So $\boldsymbol{A \neq 4}$.

So the only possibility is $A = 1, B = 3\ or\ 4, G = 2$

ie Alf always tells the truth, and Gemma always lies, with Beth telling the truth or lying at random