2007 MAT - Q5 (2 pages; 27/8/20)

Solution

(i) $f(5) = 2f(4) = 2[f(2)]^2 = 2[f(1)]^4 = 2[2f(0)]^4 = 2^5 = 32$

(ii) 4

(iii) $g(5) = 1 + g(4) = 1 + [1 + g(2)] = 2 + [1 + g(1)]$

$= 3 + [1 + g(0)] = 4$

(iv) $g(2^k) = 1 + g(0) = 1$ if $k = 0$

If $k > 0$, $g(2^k) = 1 + g(2^{k-1}) = 2 + g(2^{k-2})$

$= \cdots = k + g(2^{k-k})$

[noting that all terms are of the form $r + g(2^{k-r})$]

$= k + g(1) = k + [1 + g(0)] = k + 1$

(v) If $k = 0$, then $g(2^l + 2^k) = g(2^l + 1) = 1 + g(2^l)$

$= 1 + (l + 1) = l + 2$, from (iv)

If $k > 0$, then $g(2^l + 2^k) = g(2^k[2^{l-k} + 1])$

$= 1 + g(2^{k-1}[2^{l-k} + 1]) = \cdots = k + g(2^{l-k} + 1)$

$= k + [1 + g(2^{l-k})]$

$= k + 1 + [l - k + 1]$, from (iv)

$= l + 2$

Thus $g(2^l + 2^k) = l + 2$ in all cases.

(vi) Let RD denote the recursion depth.

From the definition of $f(n)$, for $n > 0$:

RD of $f(n) = 1 +$ RD of $f(\frac{n}{2})$

and RD of $f(n) = 1 +$ RD of $f(n-1)$

And this is the same as the definition of $g(n)$, for $n > 0$.

Also RD of $f(0) = 0$ and $g(0) = 0$