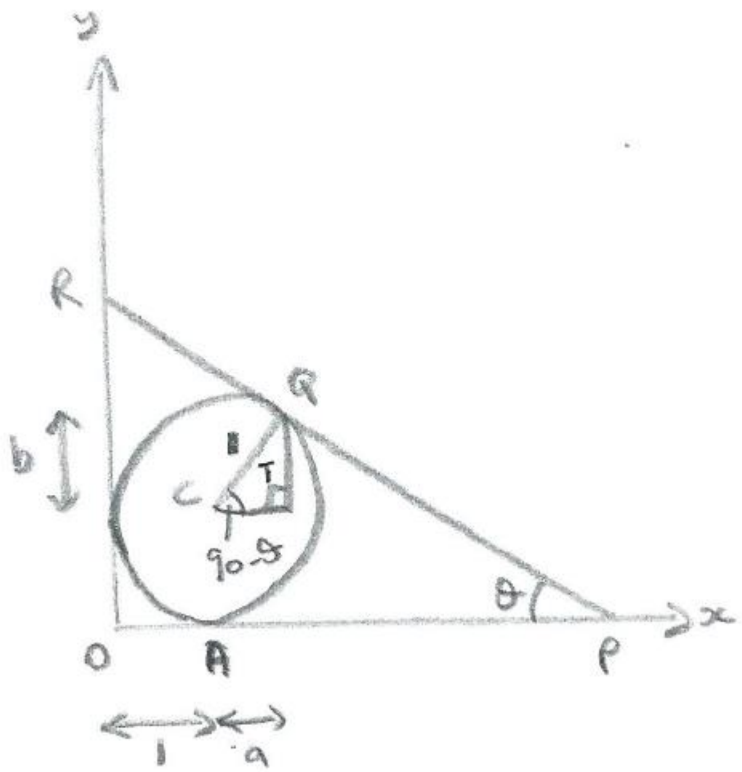
2007 MAT - Q4 (3 pages; 13/10/20)

(i)



Referring to the diagram, from triangle T:

$a = 1.sin\theta$ & $b = 1.cos\theta$

$\Rightarrow$ coordinates of Q are $(1 + a, 1 + b) = (1 + sin\theta, 1 + cos\theta)$

Gradient of PQR is $-\frac{OR}{OP} = -tan\theta$

Eq'n of PQR is $y - (1 + cos\theta) = -tan\theta(x - [1 + sin\theta])$

ie $y = 1 + cos\theta - xtan\theta + tan\theta(1 + sin\theta)$

$= \frac{1}{cos\theta}(cos\theta + cos^2\theta + sin\theta + sin^2\theta) - xtan\theta$

$= \frac{1}{cos\theta}(1 + cos\theta + sin\theta) - xtan\theta$

or $y = sec\theta + 1 + tan\theta - xtan\theta$

At P, $0 = sec\theta + 1 + tan\theta - xtan\theta$

$\Rightarrow x = \frac{sec\theta+1+tan\theta}{tan\theta} = cosec\theta + cot\theta + 1$

and so the coordinates of P are $(cosec\theta + cot\theta + 1\,, 0)$

(ii) By reversing the roles of P and R, we see that $B\left(\frac{\pi}{2} - \theta\right) = A(\theta)$.

Let the area bounded by the circle and the x & y-axes be C (which is independent of θ).

Then $C = 1 - \frac{1}{4}\pi(1)^2 = 1 - \frac{\pi}{4}$

And $A\left(\frac{\pi}{4}\right) + B\left(\frac{\pi}{4}\right) + C + \pi(1)^2 = Area(OPR)$

$\Rightarrow A\left(\frac{\pi}{4}\right) + A\left(\frac{\pi}{4}\right) + \left(1 - \frac{\pi}{4}\right) + \pi = \frac{1}{2}OP.OP$

(as $OR = OP$ when $\theta = \frac{\pi}{4}$)

$\Rightarrow 2A\left(\frac{\pi}{4}\right) + 1 + \frac{3\pi}{4} = \frac{1}{2}(cosec(\frac{\pi}{4}) + cot(\frac{\pi}{4}) + 1)^2$

$= \frac{1}{2}(\sqrt{2} + 2)^2$

So $A\left(\frac{\pi}{4}\right) = \frac{1}{4}(\sqrt{2} + 2)^2 - \frac{1}{2} - \frac{3\pi}{8}$

$= \frac{1}{2} + \sqrt{2} + 1 - \frac{1}{2} - \frac{3\pi}{8}$

$= \sqrt{2} + 1 - \frac{3\pi}{8}$

(iii) Referring to the diagram in (i), area of ACP = area of QCP,

and therefore $A(\theta) = 2 \times$ area of ACP − area of minor sector CAQ

When $\theta = \frac{\pi}{3}$, area of ACP $= \frac{1}{2}(1)\text{AP} = \frac{1}{2}(cosec\left(\frac{\pi}{3}\right) + cot\left(\frac{\pi}{3}\right))$

$= \frac{1}{2}\left(\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2}$

And area of minor sector CAQ $= \frac{\angle ACQ}{2\pi} \times \pi(1)^2 = \frac{(2\pi - \pi - \frac{\pi}{3})}{2} = \frac{\pi}{3}$

So $A\left(\frac{\pi}{3}\right) = 2\left(\frac{\sqrt{3}}{2}\right) - \frac{\pi}{3} = \sqrt{3} - \frac{\pi}{3}$, as required.