MAT - Important Ideas (17 pages; 7/10/21)

[Especially important ideas are in bold.]

Contents

(A) Tests for divisibility

(1) If the sum of the digits of a number is a multiple of 3, then the number itself is a multiple of 3; and similarly for 9.

(2) $11 \times 325847 = 3584317$

and $3 - 5 + 8 - 4 + 3 - 1 + 7 = 11$, which is a multiple of 11

This is true in all cases: If $a - b + c - d + \cdots - z$ is a multiple of 11, then $abcd \ldots z$ is a multiple of 11.

[and also for $a - b + c - d + \cdots + y$]

(B) Proof

(1) As an alternative to proving that $A \Rightarrow B$ and $B \Rightarrow A$, it may be easier to prove that $A \Rightarrow B$ and $A' \Rightarrow B'$ (as $A' \Rightarrow B'$ is equivalent to $B \Rightarrow A$).

(C) Series

(1) $\boldsymbol{\sum_{r=1}^{n} r = 1 + 2 + 3 + \cdots + n = \frac{1}{2}\, n(n + 1)}$

[Informal proof: The average size of the terms being added is

$\frac{1}{2}(1 + n)$, and there are n terms.]

(D) Factorisations

(1)**(i)** $\boldsymbol{x^2 - y^2 = (x + y)(x - y)}$

(ii) $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$

[Let $f(x) = x^3 - y^3$. Then $f(y) = 0$, and so $x - y$ is a factor of $x^3 - y^3$, by the Factor Theorem.]

$x^3 + y^3 = (x + y)(x^2 - xy + y^2)$

(iii) $x^n - y^n = (x - y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$

or $(x + y)(x^{n-1} - x^{n-2}y + \cdots + xy^{n-2} - y^{n-1})$, if n is even

$x^n + y^n = (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd

(2) Let $f(n)$ be the number of factors of n (including 1).

If $n = pq$, where p & q have no common factors (other than 1), then $f(n) = f(p)f(q)$.

[eg $100 = 2^2 \times 5^2$; factors are obtained from $\{1, 2, 4\}$ with $\{1, 5, 25\}$, giving a total of $3 \times 3 = 9$ factors: $1, 5, 25, 2, 10, 50, 4, 20, 100$]

(E) Integer solutions

eg $\boldsymbol{xy - 8x + 6y = 90}$

can be rearranged to $\boldsymbol{(x + 6)(y - 8) = 42}$

(F) Trinomial expansions

(i) $(a + b + c)^2 = (a^2 + b^2 + c^2) + 2(ab + ac + bc)$

(ii) $(a + b + c)^3 = (a^3 + b^3 + c^3)$

$+3(a^2b + a^2c + b^2a + b^2c + c^2a + c^2b)$

$+6abc$

(G) Equating coefficients

Example: To divide $f(x) = x^3 + x^2 - 11x + 10$ by $x - 2$

First of all, $f(2) = 8 + 4 - 22 + 10 = 0$, so that there is no remainder.

Then $x^3 + x^2 - 11x + 10 = (x - 2)(x^2 + ax - 5)$

Equating coefficients of x^2: $1 = a - 2$, so that $a = 3$

(Check: Equating coefficients of x: $-11 = -5 - 2a$, so that $a = 3$)

This method is usually quicker than long division.

(H) Inequalities (see Pure: "Inequalities" for further details)

(1) Beware of multiplying inequalities by a quantity that is (or could be) negative (eg $\log(0.5)$).

(2) If a and b are ≥ 0, then $a > b \Leftrightarrow a^2 > b^2$ (as $y = x^2$ is an increasing function for $x \geq 0$).

(3) If an expression can be arranged into the form $(a - b)^2$, then this will be non-negative.

(4) Methods for solving $\frac{x+1}{x-2} < 2x$

Method 1: Multiply both sides by $(x - 2)^2$ (as this is positive, assuming that $x \neq 2$). The resulting cubic will have a factor of

$x - 2$. Consider the regions of the graph.

Method 2: Treat the cases $x - 2 < 0$ and $x - 2 > 0$ separately

Method 3: Rearrange as $\frac{x+1}{x-2} - 2x \;\; < 0$, and write the LHS as a single fraction. Consider the critical points where either the numerator or the denominator is zero.

Method 4: Sketch $y = \frac{x+1}{x-2}$ and $y = 2x$, and consider the points of intersection.

(I) Logarithms

(1) $\boldsymbol{log_a b = c \;\Leftrightarrow a^c = b}$

(2) eg $\boldsymbol{3 + 2log_2 5 = 3log_2 2 + log_2(5^2)}$

$\boldsymbol{= log_2(2^3) + log_2(5^2) = log_2(8 \times 25) = log_2(200)}$

(3) $\boldsymbol{log_a b \; log_b c = log_a c}$ **or** $\boldsymbol{log_b c = \frac{log_a c}{log_a b}}$

Proof: Let $b = a^x$ & $c = b^y$

Then $c = (a^x)^y = a^{xy}$

and $log_a c = xy = log_a b \; log_b c$

Special case: $\boldsymbol{log_b c = \frac{1}{log_c b}}$

(4) As $log_8 8 = 1 \;\; and \;\; log_8 64 = 2,$ and as $y = log_8 x$ is a concave function ($\frac{dy}{dx}$ is decreasing; ie $\frac{d^2y}{dx^2} < 0$), linear interpolation

$\Rightarrow log_8\left[\frac{1}{2}(8 + 64)\right] > \frac{1}{2}(1 + 2)$

ie $log_8 36 > \frac{3}{2}$

(5) To find an upper bound for eg $\boldsymbol{log_2 3}$:

Suppose that $\boldsymbol{log_2 3 < \frac{m}{n}}$

Then $\boldsymbol{3 < 2^{\left(\frac{m}{n}\right)}}$ and $\boldsymbol{3^n < 2^m}$

As $\boldsymbol{243 = 3^5 < 2^8 = 256}$, $\boldsymbol{log_2 3 < \frac{8}{5}}$

[and $\boldsymbol{\frac{8}{5}}$ is a reasonably low upper bound, as 243 & 256 are reasonably close]

(6) eg $log_2 12 = log_2(3 \times 4) = log_2 3 + log_2 4 < \frac{8}{5} + 2 = \frac{18}{5}$,

from (5)

(7) eg $log_{36} 8 = \frac{1}{log_8 36} < \frac{2}{3}$, from (4)

(8) Example: Show that $\boldsymbol{log_5 10 < \frac{3}{2}}$

$\boldsymbol{log_5 10 < \frac{3}{2} \Leftrightarrow 10 < 5^{\left(\frac{3}{2}\right)}}$ **(as the log function is increasing)**

$\boldsymbol{\Leftrightarrow 10^2 < 5^3 \Leftrightarrow 100 < 125}$

(J) Quadratics

(1) Quadratic Functions

Example: $y = x^2 - 2x - 3$

$x^2 - 2x - 3 = (x+1)(x-3)$

Also $x^2 - 2x - 3 = (x-1)^2 - 4$

The minimum point of $(1, -4)$ lies on the line of symmetry of the curve, which is equidistant from the two roots of $x^2 - 2x - 3 = 0$: -1 & 3.

Also, from the quadratic formula (which is itself derived by completing the square on $ax^2 + bx + c$):

$x = \frac{2 \pm \sqrt{4+12}}{2} = 1 \pm 2$

Thus the roots of $x^2 - 2x - 3 = 0$ lie the same distance either side of the line of symmetry of the curve.

(2) Factorisation of quadratics

Example : $f(x) = 6x^2 + x - 12$

We need to find A and B such that $A + B = 1$ (the coefficient of x) and $AB = -72$ (the product of the coefficient of x^2 and the constant term)

$A = 9$ and $B = -8$ satisfy this

Then $f(x) = 6x^2 + 9x - 8x - 12$

$= 3x(2x+3) - 4(2x+3)$

$= (3x-4)(2x+3)$

Alternatively, $f(x) = 6x^2 - 8x + 9x - 12$

$= 2x(3x-4) + 3(3x-4)$

$= (2x+3)(3x-4)$

(K) Polynomials

(1) Integer roots

Let $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$

where $n \geq 2$ and the a_i are integers, with $a_0 \neq 0$.

Then it can be shown that any rational root of the equation $f(x) = 0$ will be an integer.

<u>Proof</u>

Suppose that there is a rational root $\frac{p}{q}$, where p & q are integers with no common factor greater than 1 and $q > 0$.

Then $\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \cdots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0$

and, multiplying by q^{n-1}:

$$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \cdots + a_1pq^{n-2} + a_1q^{n-1} = 0$$

Then, as all the terms from $a_{n-1}p^{n-1}$ onwards are integers, it follows that $\frac{p^n}{q}$ is also an integer, and hence $q = 1$ (as p & q have no common factor greater than 1), and the root is an integer.

(L) Turning Points

(1) $\frac{d^2y}{dx^2} \neq 0$ is a sufficient (but not necessary) condition for a turning point (eg $\frac{d^2y}{dx^2} = 0$ at $x = 0$ for $y = x^4$)

(2) A necessary and sufficient condition for a turning point is that the 1st non-zero derivative of the function should be of even order (and ≥ 2) (eg $y = x^4$, where $\frac{dy}{dx} = \frac{d^2y}{dx^2} = \frac{d^3y}{dx^3} = 0$, but

$\frac{d^4y}{dx^4} \neq 0$)

(3) To find the turning points of $y = \frac{x^2-2x+2}{x^2-3x-4}$, consider the quadratic $\frac{x^2-2x+2}{x^2-3x-4} = k$, with $b^2 - 4ac = 0$ (to give a quadratic in k).

(M) Greatest or least value of a function

(1) Beware of establishing the greatest or least value of a function from stationary points: these only indicate local maxima and minima.

Also, a greatest or least value may occur at a boundary of the domain.

(2) Possibilities for demonstrating that $f(x) \geq 0$

(i) $f(x) = [g(x)]^2 + [h(x)]^2$ (for all x)

(ii) For $x \geq a$: establish that $f(a) \geq 0$ and that $f'(x) \geq 0$

for $x \geq a$.

(N) Cubics

(1) Cubics always have (exactly) one point of inflexion:

$f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$

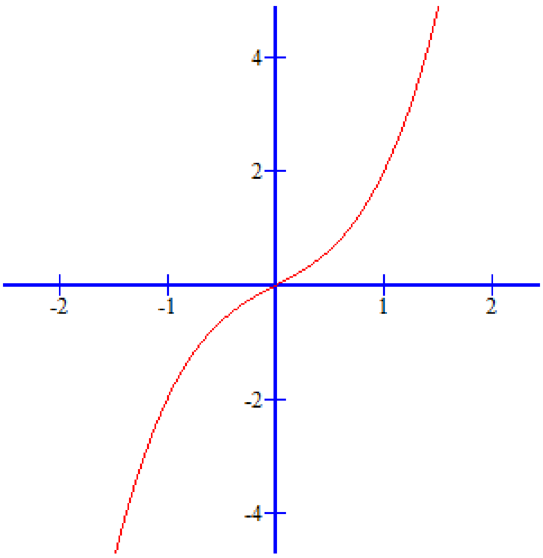
So $f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

[For a general function, $f''(x) = 0$ is a necessary (but not sufficient) condition for a point of inflexion (which is a turning point of the gradient). However, for a cubic it is a sufficient condition as well.]

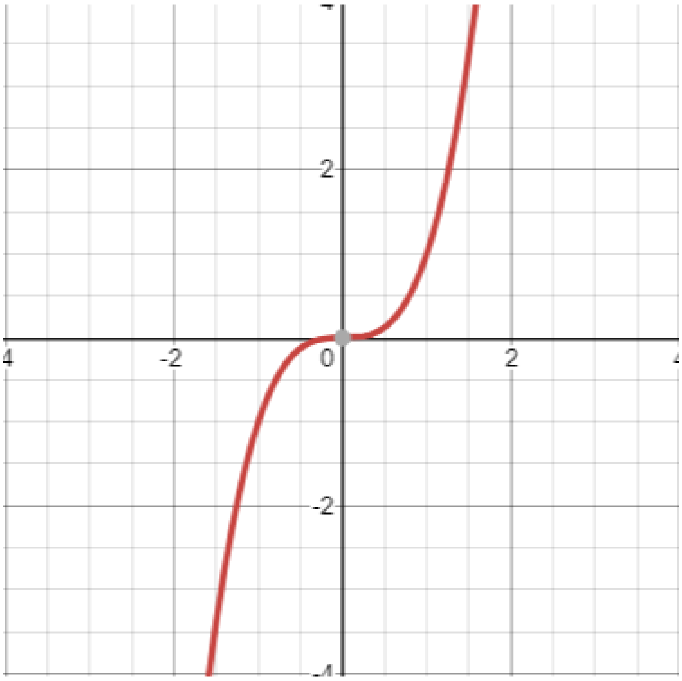
(2) There is rotational symmetry about the point of inflexion, and this implies that the point of inflexion is halfway between the turning points (if they exist).

(3) The shape of a cubic will be determined by the number of stationary points (0, 1 or 2);

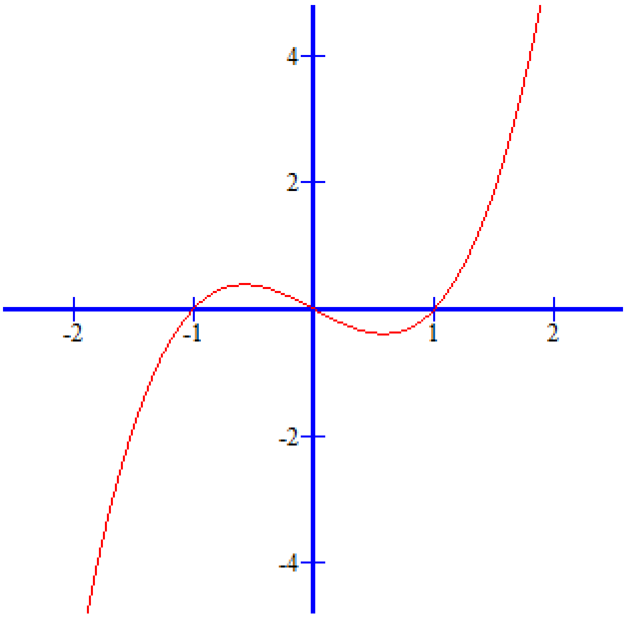
Shape 1: $y = x^3 + x$ (0 stationary points):



Shape 2: $y = x^3$ (1 stationary point)



Shape 3: $y = x^3 - x$ (2 stationary points):



(O) Transformations

(1) Translation of $\begin{pmatrix} a \\ b \end{pmatrix}$: $y = f(x) \to y - b = f(x - a)$

(2) Stretch of scale factor k in the x direction (eg if $k = 2$, graph of

$y = x^2$ is stretched outwards, so that the x-coordinates are doubled): $y = f(x) \to y = f(\frac{x}{k})$

Stretch of scale factor k in the y direction: $y = f(x) \to \frac{y}{k} = f(x)$

(3) Note that, at each stage of a composite transformation, we must be replacing x by either $x + a$ (where a can be negative) or kx (and similarly for y).

(4) Reflection in the line $x = L$: $y = f(x) \to y = f(2L - x)$

Reflection in the line $y = L$: $y = f(x) \to 2L - y = f(x)$

Special cases:

Reflection in the line $x = 0$: $f(x) \to f(-x)$

Reflection in the line $y = 0$: $y = f(x) \to -y = f(x)$

(5) Example: To obtain $y = \sin{(2x + 60)}$ from $y = sinx$,

either (a) stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x)$, and then translate by $\begin{pmatrix} -30 \\ 0 \end{pmatrix}$, to give

$y = \sin(2[x + 30]) = \sin{(2x + 60)}$

or (b) translate by $\begin{pmatrix} -60 \\ 0 \end{pmatrix}$, to give $y = \sin(x + 60)$, and then

stretch by scale factor $\frac{1}{2}$ in the x direction, to give

$y = \sin(2x + 60)$ [It is perhaps more awkward to produce a sketch by method (b).]

[Note that, at each stage, we are either replacing x by kx, or by

$x \pm a$]

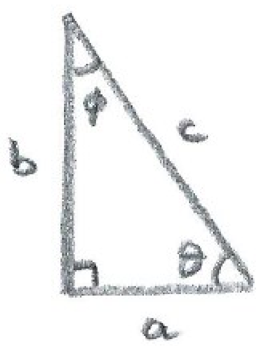
(6) A rotation of 180° is equivalent to a reflection in the line

$x = 0$, followed by a reflection in the line $y = 0$, so that $y = f(x)$

$\rightarrow y = -f(-x)$

(P) Trigonometry

(1) Relation between *sin* and *cos*



Referring to the diagram,

$sin\theta = \frac{b}{c} = cos\phi = cos\ (90° - \theta)$

and $cos\theta = \frac{a}{c} = sin\phi = sin(90° - \theta)$

(The 'co' in cosine stands for 'complementary', because θ and $90° - \theta$ are described as complementary angles.)

(2) Key Results

(A) Compound Angle formulae

$\boldsymbol{sin(\theta + \phi) = sin\theta cos\phi + cos\theta sin\phi}$

$\boldsymbol{cos(\theta + \phi) = cos\theta cos\phi - sin\theta sin\phi}$

(B) $\boldsymbol{sin(\theta \pm 360°) = sin\theta}$**;** $\boldsymbol{cos(\theta \pm 360°) = cos\theta}$

$\boldsymbol{cos(-\theta) = cos\theta}$; $\boldsymbol{sin(-\theta) = -sin\theta}$

$\boldsymbol{sin(180° - \theta) = sin\theta}$ ***;*** $\boldsymbol{cos(180° - \theta) = -cos\theta}$

$\boldsymbol{sin\theta = \cos(90° - \theta)}$ **;** $\boldsymbol{cos\theta = sin(90° - \theta)}$

(C) Translations

$\boldsymbol{sin(\theta + 90°)}$ **is** $\boldsymbol{sin\theta}$ **translated 90° to the left, which is** $\boldsymbol{cos\theta}$

$\boldsymbol{sin(\theta - 90°)}$ **is** $\boldsymbol{sin\theta}$ **translated 90° to the right, which is** $\boldsymbol{-cos\theta}$

$\boldsymbol{cos(\theta + 90°)}$ **is** $\boldsymbol{cos\theta}$ **translated 90° to the left, which is** $\boldsymbol{-sin\theta}$

$\boldsymbol{cos(\theta - 90°)}$ **is** $\boldsymbol{cos\theta}$ **translated 90° to the right, which is** $\boldsymbol{sin\theta}$

(3) To solve eg $\boldsymbol{\sin(2x - 60°) = 0.5}$; $\boldsymbol{0 \le x \le 360°}$:

Let $\boldsymbol{u = 2x - 60°}$ **and note that** $\boldsymbol{-60° \le u \le 660°}$

Having found the solutions for $\boldsymbol{u}$ **(such that** $\boldsymbol{-60° \le u \le 660°}$**), the solutions for** $\boldsymbol{x}$ **are obtained from** $\boldsymbol{x = \frac{1}{2}(u + 60)}$.

(4) Starting with $\boldsymbol{cos^2\theta + sin^2\theta = 1}$ (A) and

$\boldsymbol{cos^2\theta - sin^2\theta = cos2\theta}$ (B),

$\frac{1}{2}[(A)+(B)] \Rightarrow \boldsymbol{cos^2\theta = \frac{1}{2}(1 + cos2\theta)}$

and $\frac{1}{2}[(A)-(B)] \Rightarrow \boldsymbol{sin^2\theta = \frac{1}{2}(1 - cos2\theta)}$

(Q) Symmetry

(1) Symmetry about $\boldsymbol{x = a}$: $\boldsymbol{f(a-\lambda) = f(a+\lambda)}$ for all $\boldsymbol{\lambda}$

[Special case: symmetry about the $\boldsymbol{y}$-axis: $\boldsymbol{f(-x) = f(x)}$]

Alternatively, $\boldsymbol{f(2a-x) = f(x)}$ for all $\boldsymbol{x}$ [setting $\boldsymbol{x = a+\lambda}$]

Example: $\boldsymbol{\sin(\pi-\theta) = sin\theta}$, and the sine curve has symmetry about $\boldsymbol{\theta = \frac{\pi}{2}}$

(2) If you are asked to sketch a curve defined for $x \in [a,b]$, consider whether it might have symmetry about the mid-point $\frac{a+b}{2}$.

(R) Counting

(1) Selections

(i) Ordered selections with repetition

Number of ways of selecting r items from n, if repetitions are allowed, and order is important $= n^r$

(ii) Ordered selections without repetition

Number of ways of selecting r items from n, if repetitions are not allowed, and order is important

$= n(n-1)\ldots(n-[r-1]) = n(n-1)\ldots(n-r+1)$

[Known as a Permutation]

$P(n,r)$ or ${}^nP_r = \frac{n!}{(n-r)!} = n(n-1)\ldots(n-r+1)$

(iii) Unordered selections without repetition

Number of ways of selecting r items from n, if repetitions are not allowed, and order is not important

[Known as a Combination.]

$C(n,r)$ or ${}^nC_r$ or $\binom{n}{r} = \frac{n!}{r!(n-r)!} = \frac{n(n-1)\ldots(n-r+1)}{r!}$

[${}^nC_r$ can be obtained from ${}^nP_r = \frac{n!}{(n-r)!}$ by dividing by $r!$, to remove duplication (the ${}^nP_r$ ordered ways can be divided into groups of $r!$, containing the same items, but in a different order).]

(iv) Unordered selections with repetition

Number of ways of selecting r items from n, if repetitions are allowed, and order is not important

eg $BBCE$ selected from $ABCDEF$ $(r = 4, n = 6)$

write as $|XX|X||X|$

(| indicates that we are moving on to the next letter, and XX indicates that we are selecting 2 items from the current letter: so $|XX|X||X|$ means: move on to B (without selecting any As); then select 2 Bs; then move on to the Cs; select 1 C; move on to D, and then on to E; select 1 E; then move on to F, but select no Fs)

$=$ Number of ways of choosing r positions for the Xs,

out of the $n-1$ $|s$ and r Xs (giving a total of $n-1+r$)

$= \binom{n-1+r}{r}$