Logarithms Q4 (24/6/23)

Prove that $\int \frac{1}{x} dx = \ln|x|$ for all $x \neq 0$,

assuming that $\int \frac{1}{x} dx = \ln x$ for $x > 0$

Solution

Method 1

If $\int \frac{1}{x} dx = \ln x$ for $x > 0$, then $\frac{d}{dx}(lnx) = \frac{1}{x}$ for $x > 0$

For the case where $x < 0$:

Let $y = -x$, so that $\frac{d}{dy}(lny) = \frac{1}{y}$, as $y > 0$

[To convert back to xs:]

Hence $\frac{d}{dx}(lny).\frac{dx}{dy} = \frac{1}{(-x)}$

giving $\frac{d}{dx}(\ln[-x])(-1) = \frac{1}{(-x)}$

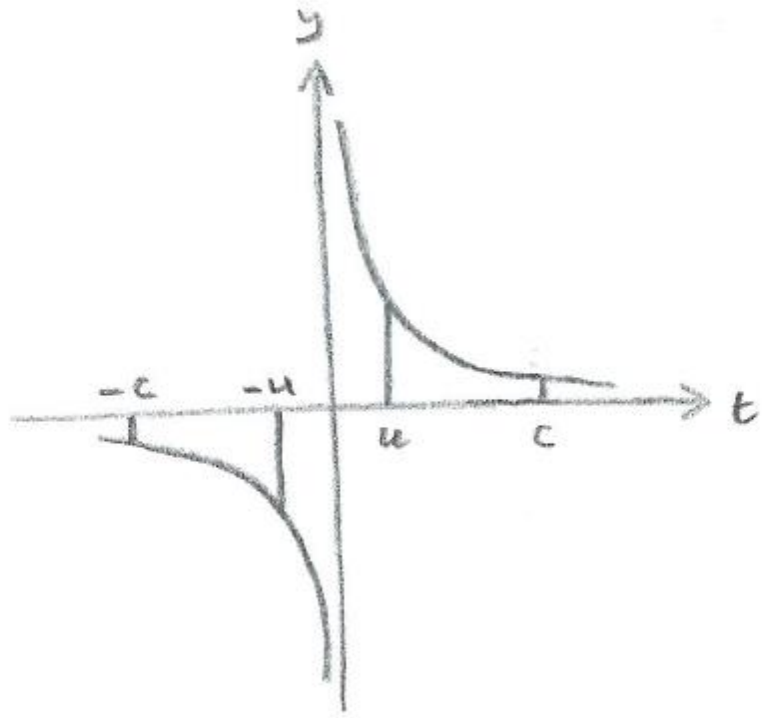
and so $\frac{d}{dx}(\ln|x|) = \frac{1}{x}$ for $x < 0$ (*)

and therefore $\int \frac{1}{x} dx = \ln|x|$ for $x < 0$, as well as $x > 0$

[Note that the function $y = \ln |x|$ for $x < 0$ is the reflection in the y-axis of $y = \ln x$ $(for\ x > 0)$, and therefore has a negative gradient, which agrees with (*).]

Method 2

Referring to the diagram below, where $u = -x > 0$ & $c > 0$,



$\int_{-c}^{x} \frac{1}{t} dt = \int_{-c}^{-u} \frac{1}{t} dt$

$=\ -$ (positive) area between graph and t-axis on LHS

$=\ -$ (positive) area between graph and t-axis on RHS

$= -\int_{u}^{c} \frac{1}{t} dt = \int_{c}^{u} \frac{1}{t} dt = lnu - lnc$

As $\int \frac{1}{x} dx$ only differs from $\int_{-c}^{x} \frac{1}{t} dt$ by an arbitrary constant, it follows that, when $x < 0$, $\int \frac{1}{x} dx = ln\, u + C = \ln|-x| + C$, as required.