Linear Programming (11 pages; 6/1/24)

Example 1

A caterer makes two types of cake: Coffee and walnut, and Victoria sponge. The table below shows the ingredients required for each cake (in tablespoons), together with the total ingredients available. The profit margins (in £) are also shown. The aim is to maximise profit, by making appropriate numbers of the two types of cake.

	Coffee and walnut	Victoria sponge	Total available
Flour	5	4	50
Sugar	2	3	30
Profit	3	2	

(i) Formulate this as a linear programming problem.

(ii) Establish the Feasible Region graphically.

(iii) Find the optimal solution by finding the value of P at each vertex of the Feasible Region.

(iv) Why might this not be a good solution, from a practical point of view?

(v) Compare the gradients of the Objective line and the constraint lines, to see which vertex of the Feasible Region is furthest away from the Origin, when the Objective line is moved.

Solution

[(i) Formulate this as a linear programming problem.]

	Coffee and walnut	Victoria sponge	Total available
Flour	5	4	50
Sugar	2	3	30
Profit	3	2	

Let C be the number of Coffee and walnut cakes, and V the number of Victoria sponge cakes.

Maximise $P = 3C + 2V$ [**objective function**]

subject to the following constraints:

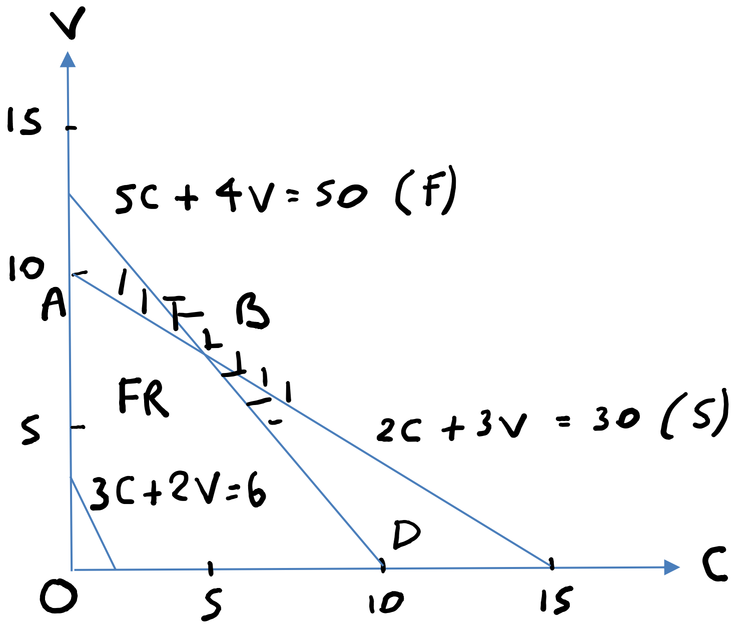
$5C + 4V \leq 50$ (flour)

$2C + 3V \leq 30$ (sugar)

$C \geq 0$, $V \geq 0$

[C & V are **control variables]**

[(ii) Establish the Feasible Region graphically.]



[The feasible region is where all the constraints are satisfied.]

Referring to the diagram, the **objective line** $P = 3C + 2V$

(or $V = \frac{P}{2} - \frac{3C}{2}$) will be parallel to $3C + 2V = 6$, and needs to be as far away from O as possible, in order to maximise P.

Unless the objective line has the same gradient as one of the constraint lines, the maximum value of P will occur at one of the vertices of the feasible region: as the objective line is moved away from O, the required vertex will be the one that the objective line crosses as it leaves the feasible region. In this example it is D.

[If the objective line does have the same gradient as one of the constraint lines, then P may be maximised along a line segment between two vertices – depending on the other constraints.]

However, if the gradients in question are similar, then it may not be easy to judge the optimal vertex from a sketch. In this case, we can simply evaluate the objective function at each vertex.

[(iii) Find the optimal solution by finding the value of P at each vertex of the Feasible Region.]

To find where the constraint lines meet:

$5C + 4V = 50$ and $2C + 3V = 30$

or $\begin{pmatrix} 5 & 4 \\ 2 & 3 \end{pmatrix}\begin{pmatrix} C \\ V \end{pmatrix} = \begin{pmatrix} 50 \\ 30 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} C \\ V \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 3 & -4 \\ -2 & 5 \end{pmatrix}\begin{pmatrix} 50 \\ 30 \end{pmatrix} = \frac{1}{7}\begin{pmatrix} 30 \\ 50 \end{pmatrix}$

So $C = \frac{30}{7}$, $V = \frac{50}{7}$

(although integer values will be required)

Then $P = 3C + 2V = \frac{190}{7} = 27\frac{1}{7}$

At $(0,10), P = 20,$

and at $(10,0), P = 30$

So the optimal solution is $P = 30$ when $C = 10$ and $V = 0$.

[(iv) Why might this not be a good solution, from a practical point of view?]

- customers may want Victoria Sponges

- situation could change (so don't lose ability to produce Victoria Sponges)

[(v) Compare the gradients of the Objective line and the constraint lines, to see which vertex of the Feasible Region is furthest away from the Origin, when the Objective line is moved.]

(F) $5C + 4V = 50$: gradient is $-\frac{5}{4}$

(S) $2C + 3V = 30$: gradient is $-\frac{2}{3}$

(P) $P = 3C + 2V$: gradient is $-\frac{3}{2}$

Line S is the 1st constraint line that P comes into contact with.

P has a steeper gradient than S, so that P first comes into contact with S at the top end, and so the vertex B will be further away from O than A, as P is moved away from O.

So B is preferred to S.

Beyond B, F is the critical constraint line.

P has a steeper gradient than F, so that vertex D will be further away from O than B, and therefore preferred to B.

Thus, D is the optimal vertex.

[Provided that the gradients are not too similar, the above approach can be performed by inspection of the sketch, by comparing the objective line with a contraint line, and seeing at which end they converge. The required vertex will then lie toward the end opposite the point of convergence.]

Example 2

Consider the following LP problem:

Minimise $\mathrm{P} = 2x + y$

subject to $y \geq 3$

$3x + 4y \geq 24$

$y \leq 3x$

$x \geq 0\ ,\ y \geq 0$

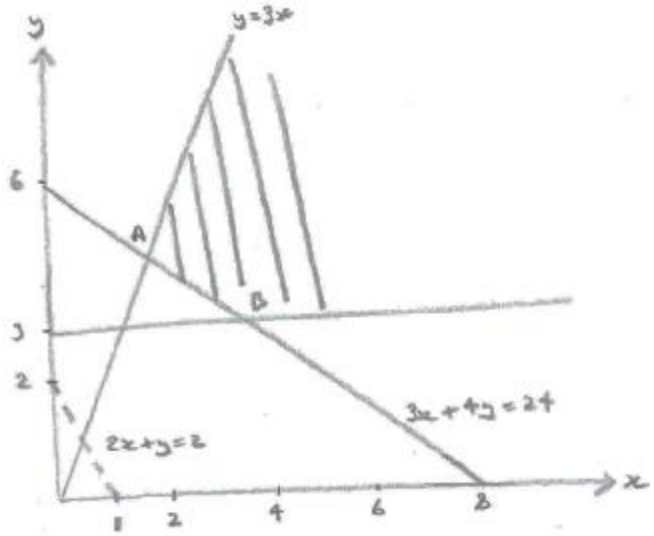
(i) Establish the Feasible Region graphically.

(ii) Find the optimal solution, assuming initially that non integer-valued solutions are acceptable.

(iii) Find a suitable integer-valued solution.

Solution

[(i) Establish the Feasible Region graphically.]



[(ii) Find the optimal solution, assuming initially that non integer-valued solutions are acceptable.]

By considering lines parallel to $2x + y = 2$, we can see that P is minimised at A.

$3x + 4y = 24$ & $y = 3x \Rightarrow 15x = 24 \Rightarrow x = \frac{8}{5} = 1.6$; thus A is (1.6,4.8), where P = 8

[(iii) Find a suitable integer-valued solution.]

If only integer solutions are acceptable, we can now consider the cases $x = 1$ and $x = 2$:

When $x = 1$, the objective function (to be minimised) is

$P = 2(1) + y$, and this means that we need to minimise y.

The constraints become

$y \geq 3$

$3(1) + 4y \geq 24$

$y \leq 3(1)$

The 1st and 3rd constraints mean that $y = 3$, but then the 2nd constraint isn't satisfied. So $x = 1$ isn't possible.

Investigating $x = 2$ instead:

When $x = 2$, the objective function (to be minimised) is

$P = 2(2) + y$, and once again we need to minimise y.

The constraints become

$y \geq 3$

$3(2) + 4y \geq 24$

$y \leq 3(2)$

y is then minimised when it equals 5

So the required integer-valued solution is $(2,5)$, when

$P = 2(2) + 5 = 9$

Notes

(i) The above approach is not guaranteed to find the best integer-valued solution: in some cases, this will lie further from the initial optimal solution (ie where non-integer values are allowed).

(ii) For information, there is a more complicated method called the Branch and Bound method, which will always find the best integer-valued solution (this is part of the OCR syllabus).

Example of a more complicated constraint

A company manufactures 3 liquid products: X, Y and Z, sold in drums. There is enough of a constituent chemical to make 45 drums of X, or 60 drums of Y, or 90 drums of Z. Formulate this constraint as an inequality.

Solution

[Assuming that it is intended to be able to produce more than one of X, Y and Z:]

If only X is being produced, then the constraint can be written as

$\frac{X}{45} \leq 1$

If (say) up to half of X's quota is to be used, and up to half of Y's, then the constraint would be $\frac{X}{45} \leq 0.5$ & $\frac{Y}{60} \leq 0.5$;

or if X is favoured over Y, eg $\frac{X}{45} \leq 0.7$ & $\frac{Y}{60} \leq 0.3$

The most flexible constraint involving all 3 products would be

$\frac{X}{45} + \frac{Y}{60} + \frac{Z}{90} \leq 1$