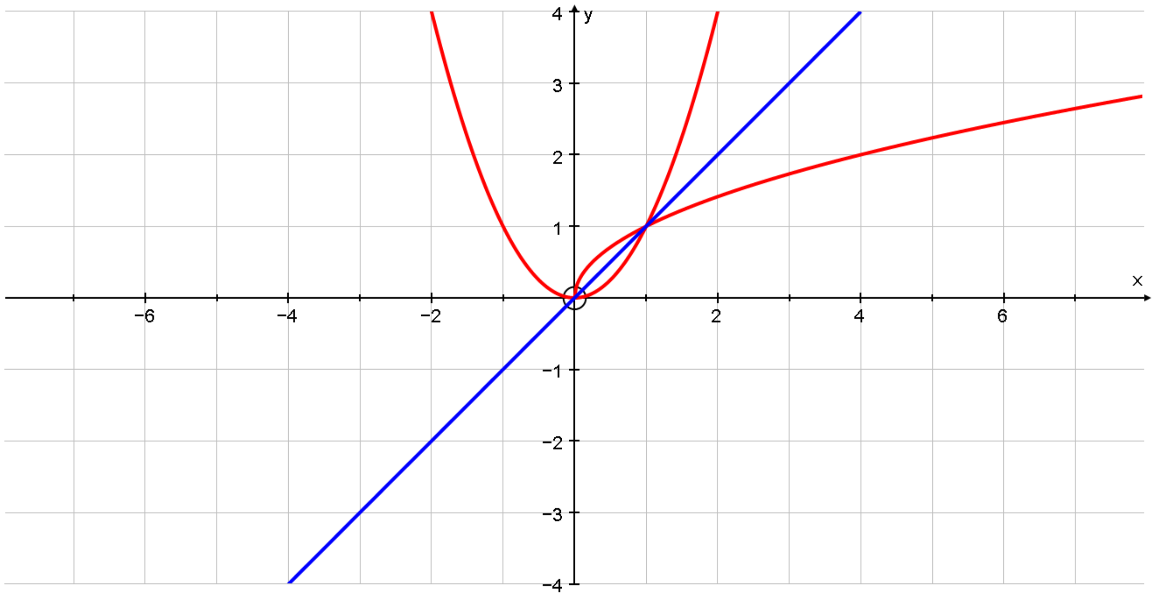
Inverse Functions (3 pages; 2/6/23)

(1) Example: $y = x^2$



There are two possible approaches:

(a) Obtain the inverse function algebraically:

(i) Make x the subject of the equation, so that $x = \sqrt{y}$

(only the positive root is used, in order for the inverse to be a function; the domain of $y = x^2$ can be restricted to $x \geq 0$ in order to achieve this)

(ii) Swap the roles of x and y (in order for the new function to have x values on the horizontal axis), to give $y = \sqrt{x}$

or (b) Reflect the function $y = x^2$ in the line $y = x$.

The point (a, a^2) then moves to (a^2, a), to obtain the inverse mapping: the y coordinate is now the square root of the x coordinate.

Alternatively, the equivalent transformation of reflecting in the y axis (producing no change in this case) and then rotating clockwise through 90° may be easier to visualise.

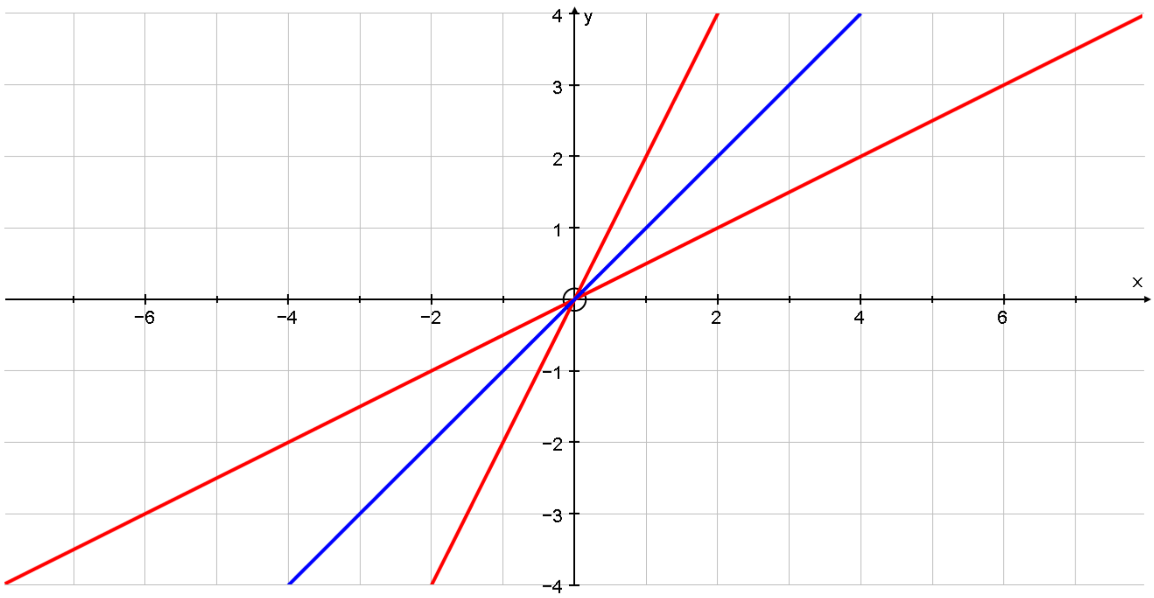
Note that $y = f(x)$ and $y = f^{-1}(x)$ will meet on the line $y = x$, if they intersect.

Also, the x and y axes need to have the same scale, in order for

$y = f^{-1}(x)$ to be the reflection of $y = f(x)$ in the x axis.

(2) Differentiating an inverse function

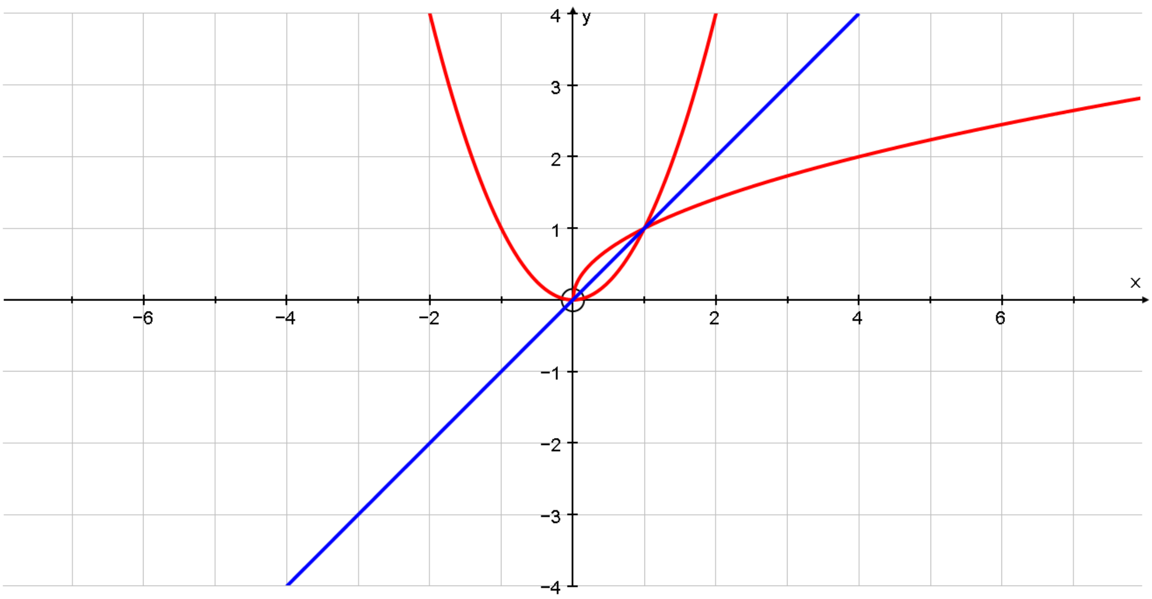
Consider first of all the simple example $y = 2x$:



The inverse function is $y = \frac{x}{2}$

The gradient of the inverse function is the reciprocal of the gradient of the original function; though in this example it is a constant gradient.

For a more complicated function such as $y = x^2$, the gradient of the inverse function $y = \sqrt{x}$ at the point (4,2), for example, will equal the reciprocal of the gradient of $y = x^2$ at the point (2,4), since (2,4) is the reflection of (4,2) in the line $y = x$.



In general, $\frac{d}{dx}f^{-1}(x)|x = b \;=\; \frac{1}{\frac{d}{dx}f(x)|x=a}$, where $f(a) = b$

(This can also be written as $\frac{dx}{dy}\;|y = b \;=\; \frac{1}{\frac{dy}{dx}\,|x=a}$)