

## Integration Exercises - Part 2 (Sol'ns)(23 pages; 6/4/21)

(Note: The constant of integration has been omitted throughout.)

$$(1) \int \frac{1}{1+e^x} dx$$

### Solution

Step 1: rearrange

$$I = \int \frac{e^{-x}}{e^{-x}+1} dx$$

Step 2: substitute  $u = e^{-x} + 1$ , so that  $du = -e^{-x} dx$

$$I = \int -\frac{1}{u} du = -\ln u = -\ln(e^{-x} + 1)$$

$$(2) \int e^{2x}(1 + e^x)^{\frac{1}{2}} dx$$

### Solution

'Tidying-up' substitution:

Let  $u = 1 + e^x$ , so that  $du = e^x dx$

$$I = \int (u - 1)^2 u^{\frac{1}{2}} \cdot \frac{1}{u-1} du$$

$$= \int u^{\frac{3}{2}} - u^{\frac{1}{2}} du$$

$$= \frac{2}{5} u^{\frac{5}{2}} - \frac{2}{3} u^{\frac{3}{2}}$$

$$= \frac{2}{15} (1 + e^x)^{\frac{3}{2}} (3(1 + e^x) - 5)$$

$$= \frac{2}{15} (1 + e^x)^{\frac{3}{2}} (3e^x - 2)$$

$$(3) \int \frac{1}{\sqrt{x}(1+\sqrt{x})} dx$$

**Solution**

Note:  $\frac{1}{\sqrt{x}}$  integrates to a multiple of  $\sqrt{x}$  (or  $1+\sqrt{x}$ )

Let  $u = 1 + \sqrt{x}$ , so that  $du = \frac{1}{2}x^{-\frac{1}{2}} dx$

$$I = 2 \int \frac{1}{u} du$$

$$= 2 \ln u = 2 \ln(1 + \sqrt{x})$$

$$(4) \int \sec x dx$$

**Solution****Method 1**

Note: This is an odd power of  $\cos x$

$$I = \int \frac{\cos x}{\cos^2 x} dx$$

$$= \int \frac{\cos x}{1 - \sin^2 x} dx$$

Let  $u = \sin x$ , so that  $du = \cos x dx$

$$\int \frac{\cos x}{1 - \sin^2 x} dx = \int \frac{1}{1 - u^2} du$$

$$\frac{1}{2} \int \frac{1}{1 - u} + \frac{1}{1 + u} du$$

$$= \frac{1}{2} \{-\ln |1 - u| + \ln |1 + u|\}$$

$$= \frac{1}{2} \ln \left| \frac{1 + \sin x}{1 - \sin x} \right|$$

$$= \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{(1 - \sin x)(1 + \sin x)} \right|$$

$$= \frac{1}{2} \ln \left| \frac{(1 + \sin x)^2}{1 - \sin^2 x} \right|$$

$$= \ln \left| \frac{1+\sin x}{\cos x} \right|$$

$$= \ln |\sec x + \tan x|$$

### Method 2

Let  $t = \tan\left(\frac{x}{2}\right)$ , so that  $\tan x = \tan\left(\frac{x}{2} + \frac{x}{2}\right) = \frac{2t}{1-t^2}$

Then it can be shown that  $2t$ ,  $1-t^2$  &  $1+t^2$  form the sides of a right-angled triangle, so that  $\sec x = \frac{1+t^2}{1-t^2}$

Also,  $dt = \sec^2\left(\frac{x}{2}\right)\left(\frac{1}{2}\right) dx = \frac{1}{2}(1+t^2)dx$

Then  $I = \int \frac{1+t^2}{1-t^2} \left(\frac{2}{1+t^2}\right) dt = 2 \int \frac{1}{1-t^2} dt = \int \frac{1}{1-t} + \frac{1}{1+t} dt$

$$= \ln \left| \frac{1+t}{1-t} \right| = \ln \left| \frac{(1+t)^2}{1-t^2} \right|$$

$$= \ln \left| \frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2} \right| = \ln |\sec x + \tan x|$$

(5)  $\int \frac{(\ln x)^2}{x} dx$

### Solution

Noting that  $\frac{1}{x}$  integrates to  $\ln x$ ,

let  $u = \ln x$ , so that  $du = \frac{1}{x} dx$

$$I = \int u^2 du = \frac{1}{3} u^3 = \frac{1}{3} (\ln x)^3$$

(6)  $\int \frac{x^5}{4-x^3} dx$

**Solution**

Let  $u = 4 - x^3$ , so that  $du = -3x^2 dx$

[A slightly speculative 'tidying-up' substitution.]

$$\begin{aligned}
 I &= -\frac{1}{3} \int \frac{x^3 du}{u} \\
 &= -\frac{1}{3} \int \frac{4-u}{u} du \\
 &= -\frac{1}{3} \int \frac{4}{u} - 1 du \\
 &= -\frac{1}{3} (4 \ln |u| - u) \\
 &= -\frac{1}{3} (4 \ln |4 - x^3| + x^3 - 4)
 \end{aligned}$$

or  $-\frac{1}{3} (4 \ln |4 - x^3| + x^3)$ , as the 4 can be included in the constant of integration

**Alternative method**

$$\begin{aligned}
 \int \frac{x^5}{4-x^3} dx &= \int \frac{x^5-4x^2}{4-x^3} + \frac{4x^2}{4-x^3} dx \\
 &= \int -x^2 dx + \frac{4}{3} \int \frac{3x^2}{4-x^3} dx
 \end{aligned}$$

For the 2nd integral (J), let  $u = x^3$ , so that  $du = 3x^2 dx$

$$\text{and } J = \frac{4}{3} \int \frac{1}{4-u} du = -\frac{4}{3} \ln |4 - x^3|,$$

$$\text{so that } I = -\frac{1}{3} x^3 - \frac{4}{3} \ln |4 - x^3|$$

$$(7) \int \sqrt{1 + \sin 2x} dx$$

**Solution**

$$1 + \sin 2x = 1 + \cos\left(\frac{\pi}{2} - 2x\right) = 1 + \cos 2\left(\frac{\pi}{4} - x\right) = 2\cos^2\left(\frac{\pi}{4} - x\right)$$

so that  $\sqrt{1 + \sin 2x} = \sqrt{2} \cos\left(\frac{\pi}{4} - x\right)$  and  $I$  can be found easily.

$$(8) \int \arctan x \, dx$$

**Solution**

Applying Parts; integrating 1:

$$I = x \arctan x - \int \frac{x}{x^2+1} \, dx = x \arctan x - \frac{1}{2} \ln(x^2 + 1)$$

$$(9) \int \frac{1}{x \ln x} \, dx$$

**Solution**

$$= \int \frac{\left(\frac{1}{x}\right)}{\ln x} \, dx, \text{ and as } \int \frac{1}{x} \, dx = \ln x, \text{ let } u = \ln x, \text{ so that } du = \frac{1}{x} \, dx$$

$$\text{Then } I = \int \frac{1}{u} \, du = \ln(\ln x)$$

If we try Parts, we get:

$$I = \ln x \left(\frac{1}{\ln x}\right) - \int \ln x (-1)(\ln x)^{-2} \left(\frac{1}{x}\right) \, dx$$

$$= 1 + \int \frac{1}{x \ln x} \, dx = 1 + I$$

This isn't a contradiction, because we haven't allowed for the arbitrary constant. However the method doesn't give us an answer.

$$(10) \int \frac{e^x}{e^{2x}+1} dx$$

### Solution

As  $\int e^x dx = e^x$ , and the rest of the integrand is a function of  $e^x$  that we can integrate, let  $u = e^x$ , so that  $du = e^x dx$

$$\text{Then } I = \int \frac{1}{u^2+1} du = \arctan(e^x)$$

$$(11) \int \frac{\sec^2 x}{4+\tan^2 x} dx$$

### Solution

As  $\int \sec^2 x dx = \tan x$ , let  $u = \tan x$ , so that  $du = \sec^2 x dx$ ,

$$\text{and } I = \int \frac{1}{4+u^2} du = \frac{1}{2} \arctan\left(\frac{\tan x}{2}\right)$$

$$(12) \int \frac{1}{1-\sin x} dx$$

### Solution

$$I = \int \frac{1+\sin x}{1-\sin^2 x} dx = \int \frac{1+\sin x}{\cos^2 x} dx$$

$$= \int \sec^2 x dx + J = \tan x + J$$

For  $J$ : let  $u = \cos x$ , so that  $du = -\sin x dx$

$$\text{and } J = -\int u^{-2} du = u^{-1} = \sec x$$

$$\text{So } I = \tan x + \sec x$$

$$(13) \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4x^2+9} dx$$

**Solution**

**Method 1**

$$\begin{aligned} I &= \frac{1}{4} \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{x^2 + \left(\frac{3}{2}\right)^2} dx = \frac{1}{4} \left[ \frac{1}{\left(\frac{3}{2}\right)} \arctan \left( \frac{x}{\left(\frac{3}{2}\right)} \right) \right]_{-\frac{3}{2}}^{\frac{3}{2}} \\ &= \frac{1}{6} (\arctan(1) - \arctan(-1)) = \frac{1}{6} \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) \\ &= \frac{\pi}{12} \end{aligned}$$

**Method 2**

$$\begin{aligned} \text{Let } u &= 2x, \text{ so that } du = 2dx \text{ and } I = \int_{-3}^3 \frac{1}{u^2+9} \left(\frac{1}{2}\right) du \\ &= \frac{1}{2} \left[ \left(\frac{1}{3}\right) \arctan \left(\frac{u}{3}\right) \right]_{-3}^3 = \frac{1}{6} (\arctan(1) - \arctan(-1)) \\ &= \frac{1}{6} \left( \frac{\pi}{4} - \left( -\frac{\pi}{4} \right) \right) = \frac{\pi}{12} \end{aligned}$$

$$(14) \int \frac{x}{1+x^4} dx$$

**Solution**

As  $\int x dx = \frac{1}{2}x^2$ , and  $\frac{1}{1+u^2}$  can be integrated,

let  $u = x^2$ , so that  $du = 2x dx$ ,

$$\text{and } I = \frac{1}{2} \int \frac{1}{1+u^2} du = \frac{1}{2} \arctan(x^2)$$

$$(15) \int \frac{\sin 2x}{1 + \cos x} dx$$

### Solution

As a (speculative) simplifying substitution, let  $u = 1 + \cos x$ ,

so that  $du = -\sin x dx$

$$\text{Then } I = 2 \int \frac{\sin x \cos x}{u} \left( -\frac{du}{\sin x} \right) = -2 \int \frac{u-1}{u} du$$

$$= -2(u - \ln u) = 2(\ln(1 + \cos x) - \cos x - 1)$$

or  $2(\ln(1 + \cos x) - \cos x)$ ,

including the  $-2$  in the constant of integration

$$(16)^* \int_0^1 \sqrt{16x^2 + 9} dx$$

### Solution

Let  $4x = 3 \tan \theta$ , so that  $4dx = 3 \sec^2 \theta d\theta$

$$\Rightarrow I = \int_0^{\arctan(\frac{4}{3})} \sqrt{9 \tan^2 \theta + 9} \left( \frac{3}{4} \right) \sec^2 \theta d\theta$$

$$= \frac{9}{4} \int_0^{\arctan(\frac{4}{3})} \sec^3 \theta d\theta$$

To find  $J = \int \sec^3 \theta d\theta$ :

#### Method 1 (Parts - speculative)

$$J = \int \sec^2 \theta \sec \theta d\theta$$

Integrating  $\sec^2 \theta$  and differentiating  $\sec \theta = (\cos \theta)^{-1}$ :

$$J = \tan \theta \sec \theta - \int \tan \theta (-1)(\cos \theta)^{-2} (-\sin \theta) d\theta$$

$$= \tan \theta \sec \theta - \int \tan^2 \theta \sec \theta d\theta$$

$$= \tan \theta \sec \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta$$



$$= \tan\theta \sec\theta - J + \int \sec\theta d\theta$$

$$\Rightarrow 2J = \tan\theta \sec\theta + \ln |\sec\theta + \tan\theta|$$

[from the formulae booklet]

$$\text{So } I = \frac{9}{8} [\tan\theta \sec\theta + \ln |\sec\theta + \tan\theta|]_{\arctan(\frac{4}{3})}^0$$

$$\theta = \arctan\left(\frac{4}{3}\right) \Rightarrow \tan\theta = \frac{4}{3} \Rightarrow \sec\theta = \frac{5}{3} \text{ (from a 3,4,5 triangle)}$$

$$\text{Then } I = \frac{9}{8} \left\{ \left(\frac{4}{3}\right) \left(\frac{5}{3}\right) + \ln \left| \frac{5}{3} + \frac{4}{3} \right| - 0 - \ln|1| \right\}$$

$$= \frac{9}{8} \left\{ \frac{20}{9} + \ln 3 \right\} = \frac{5}{2} + \frac{9}{8} \ln 3$$

## Method 2

$$J = \int \frac{1}{\cos^3\theta} d\theta = \int \frac{\cos\theta}{(1-\sin^2\theta)^2} d\theta$$

Let  $u = \sin\theta$ , so that  $du = \cos\theta d\theta$

$$\text{and } J = \int \frac{1}{(1-u^2)^2} du$$

$$\frac{1}{(1-u^2)^2} = \frac{1}{(1-u)^2(1+u)^2} = \frac{A}{1-u} + \frac{B}{(1-u)^2} + \frac{C}{1+u} + \frac{D}{(1+u)^2}$$

$$\text{so that } 1 = A(1-u)(1+u)^2 + B(1+u)^2 + C(1+u)(1-u)^2 + D(1-u)^2$$

$$\text{Then } u = 1 \Rightarrow 1 = 4B \Rightarrow B = \frac{1}{4}$$

$$\text{and } u = -1 \Rightarrow 1 = 4D \Rightarrow D = \frac{1}{4}$$

$$u = 0 \Rightarrow 1 = A + B + C + D$$

$$\text{Equating coeffs of } u^3: 0 = -A + C$$

$$\text{Hence } 1 = A + \frac{1}{4} + A + \frac{1}{4} \text{ and so } A = C = \frac{1}{4}$$

$$\begin{aligned}
\text{Therefore } J &= \frac{1}{4} \int \frac{1}{1-u} + \frac{1}{(1-u)^2} + \frac{1}{1+u} + \frac{1}{(1+u)^2} du \\
&= \frac{1}{4} (-\ln|1-u| + (1-u)^{-1} + \ln|1+u| - (1+u)^{-1}) \\
&= \frac{1}{4} \left( \ln \left| \frac{1+u}{1-u} \right| + \frac{(1+u)-(1-u)}{(1-u)(1+u)} \right) \\
&= \frac{1}{4} \left( \ln \left| \frac{1+u}{1-u} \right| + \frac{2u}{1-u^2} \right) \\
&= \frac{1}{4} \left( \ln \left| \frac{1+\sin\theta}{1-\sin\theta} \right| + \frac{2\sin\theta}{\cos^2\theta} \right) \\
&= \frac{1}{4} \left( \ln \left| \frac{(1+\sin\theta)^2}{(1-\sin\theta)(1+\sin\theta)} \right| + 2\tan\theta\sec\theta \right) \\
&= \frac{1}{4} \left( \ln \left| \frac{(1+\sin\theta)^2}{\cos^2\theta} \right| + 2\tan\theta\sec\theta \right) \\
&= \frac{1}{2} \left( \ln \left| \frac{1+\sin\theta}{\cos\theta} \right| + \tan\theta\sec\theta \right) \\
&= \frac{1}{2} (\ln|\sec\theta + \tan\theta| + \tan\theta\sec\theta)
\end{aligned}$$

(then as before)

$$(17) \int \sec^3 x \, dx$$

**Solution**

See (16)

$$(18) \int \sec^4 x \, dx$$

**Solution**

$$I = \int \sec^2 x (1 + \tan^2 x) \, dx = \tan x + \int \sec^2 x \tan^2 x \, dx$$

Then let  $u = \tan x$ , so that  $du = \sec^2 x \, dx$ ,

$$\text{and } I = \tan x + \int u^2 du = \tan x + \frac{1}{3} \tan^3 x$$

$$(19) \int \cos^5 x dx$$

**Solution**

$$I = \int \cos x (1 - \sin^2 x)^2 dx$$

$$= \int \cos x dx - 2 \int \cos x \sin^2 x dx + \int \cos x \sin^4 x dx$$

Let  $u = \sin x$ , so that  $du = \cos x dx$

$$\text{Then } I = \sin x - 2 \int u^2 du + \int u^4 du$$

$$= \sin x - \frac{2}{3} \sin^3 x + \frac{1}{5} \sin^5 x$$

$$(20) \int \cos x \ln(\cos x) dx$$

**Solution**

By Parts, integrating  $\cos x$  & differentiating  $\ln(\cos x)$ :

$$I = \sin x \ln(\cos x) - \int \sin x \left( \frac{1}{\cos x} \right) (-\sin x) dx$$

$$= \sin x \ln(\cos x) + \int (1 - \cos^2 x) \sec x dx$$

$$= \sin x \ln(\cos x) + J,$$

$$\text{where } J = \int \sec x dx - \int \cos x dx$$

$$= \ln|\sec x + \tan x| - \sin x$$

$$\text{So } I = \sin x \ln(\cos x) + \ln|\sec x + \tan x| - \sin x$$

$$(21) \int \tan x \sin x \, dx$$

**Solution**

$$\begin{aligned} I &= \int \frac{\sin^2 x}{\cos x} \, dx = \int (1 - \cos^2 x) \sec x \, dx \\ &= \int \sec x \, dx - \int \cos x \, dx = \ln|\sec x + \tan x| - \sin x \end{aligned}$$

$$(22) \int \sin 3x \cos x \, dx$$

**Solution**

**Method 1**

$$\sin 4x = \sin(3x + x) = \sin 3x \cos x + \cos 3x \sin x$$

$$\& \sin 2x = \sin(3x - x) = \sin 3x \cos x - \cos 3x \sin x,$$

$$\text{so that } \sin 3x \cos x = \frac{1}{2}(\sin 4x + \sin 2x)$$

$$\begin{aligned} \text{and } I &= \frac{1}{2} \int \sin 4x + \sin 2x \, dx = \frac{1}{2} \left( -\frac{1}{4} \cos 4x - \frac{1}{2} \cos 2x \right) \\ &= -\frac{1}{8} (\cos 4x + 2 \cos 2x) \end{aligned}$$

**Method 2**

$$\sin 3x = \sin(2x + x) = \sin 2x \cos x + \cos 2x \sin x$$

$$= 2 \sin x \cos^2 x + (\cos^2 x - \sin^2 x) \sin x$$

$$= 3 \sin x \cos^2 x - \sin^3 x$$

$$\text{so that } I = \int 3 \sin x \cos^3 x \, dx - \int \sin^3 x \cos x \, dx$$

Let  $u = \cos x$  for the 1st integral, so that  $du = -\sin x \, dx$ ,

and let  $v = \sin x$  for the 2nd integral, so that  $dv = \cos x \, dx$

$$\text{Then } I = -3 \int u^3 \, du - \int v^3 \, dv$$

$$= -\frac{3}{4}\cos^4 x - \frac{1}{4}\sin^4 x$$

$$= -\frac{1}{4}(3\cos^4 x + \sin^4 x)$$

[As a check, we can substitute values of  $x$  such as  $0$  &  $\frac{\pi}{2}$ , to compare the answers by the two methods. There is in fact a difference of  $\frac{3}{8}$  in each case, but this is due to the constant of integration.]

### Method 3

By Parts, integrating  $\cos x$  and differentiating  $\sin 3x$  (for example):

$$I = \sin x \sin 3x - \int \sin x (3\cos 3x) dx$$

Then integrating  $\sin x$  and differentiating  $\cos 3x$ ,

$$I = \sin x \sin 3x - 3\{-\cos x \cos 3x - \int (-\cos x)(-3\sin 3x) dx\}$$

$$= \sin x \sin 3x + 3\cos x \cos 3x + 9I$$

$$\text{and hence } I = -\frac{1}{8}(\sin x \sin 3x + 3\cos x \cos 3x)$$

[Substituting values of  $x$  shows agreement with Method 1.]

$$(23) \int \operatorname{cosec} x \, dx$$

### Solution

#### Method 1

$$I = \int \frac{1}{\sin x} \, dx = \int \frac{\sin x}{\sin^2 x} \, dx = \int \frac{\sin x}{1 - \cos^2 x} \, dx$$

Let  $u = \cos x$ , so that  $du = -\sin x \, dx$

$$\text{Then } I = -\int \frac{1}{1-u^2} \, du = -\frac{1}{2} \int \frac{1}{1-u} + \frac{1}{1+u} \, du$$

$$= -\frac{1}{2} (\ln|1+u| - \ln|1-u|)$$

$$= -\frac{1}{2} \ln \left| \frac{1+\cos x}{1-\cos x} \right|$$

#### Method 2

$$\text{Let } t = \tan\left(\frac{x}{2}\right), \text{ so that } \tan x = \tan\left(\frac{x}{2} + \frac{x}{2}\right) = \frac{2t}{1-t^2}$$

Then it can be shown that  $2t$ ,  $1-t^2$  &  $1+t^2$  form the sides of a right-angled triangle, so that  $\operatorname{cosec} x = \frac{1+t^2}{2t}$

$$\text{Also, } dt = \sec^2\left(\frac{x}{2}\right) \left(\frac{1}{2}\right) dx = \frac{1}{2}(1+t^2)dx$$

$$\text{so that } I = \int \frac{1+t^2}{2t} \left(\frac{2}{1+t^2}\right) dt = \int \frac{1}{t} dt = \ln \left| \tan\left(\frac{x}{2}\right) \right|$$

[To reconcile these two answers:

$$\text{rtp (result to prove): } \ln \left| \frac{1-\cos x}{1+\cos x} \right| = 2 \ln \left| \tan\left(\frac{x}{2}\right) \right|$$

$$\Leftrightarrow \frac{1-\cos x}{1+\cos x} = \tan^2\left(\frac{x}{2}\right)$$

$$LHS = \frac{1 - \frac{1-t^2}{1+t^2}}{1 + \frac{1-t^2}{1+t^2}} = \frac{(1+t^2) - (1-t^2)}{(1+t^2) + (1-t^2)} = t^2 = RHS ]$$

$$(24) \int \tan^3 x \, dx$$

**Solution**

$$I = \int (\sec^2 x - 1) \tan x \, dx = \int \sec^2 x \tan x \, dx - \int \tan x \, dx$$

For the 1st integral,  $\int \sec^2 x \, dx = \tan x$ , so let  $u = \tan x$

Then  $du = \sec^2 x \, dx$  and  $I = \int u \, du - \ln|\sec x|$

$$= \frac{1}{2} \tan^2 x - \ln|\sec x|$$

$$(25) \int \tan^4 x \, dx$$

**Solution**

$$I = \int (\sec^2 x - 1) \tan^2 x \, dx$$

For the 1st integral,  $\int \sec^2 x \, dx = \tan x$ , so let  $u = \tan x$

Then  $du = \sec^2 x \, dx$  and  $I = \int u^2 \, du - \int \sec^2 x - 1 \, dx$

$$= \frac{1}{3} \tan^3 x - \tan x + x$$

$$(26) \int \frac{\cos^3 x}{\sin^2 x} \, dx$$

**Solution**

$$I = \int \frac{\cos x (1 - \sin^2 x)}{\sin^2 x} \, dx$$

Let  $u = \sin x$ , so that  $du = \cos x \, dx$ ,

$$\text{and } I = \int \frac{1-u^2}{u^2} \, du = \int u^{-2} - 1 \, du = -u^{-1} - u = -\operatorname{cosec} x - \sin x$$

$$(27) \int \frac{\cos x + \sin x}{\cos x - \sin x} dx$$

### Solution

As  $\int \cos x + \sin x dx = \sin x - \cos x$ ,

let  $u = \sin x - \cos x$ , so that  $du = \cos x + \sin x dx$ ,

and  $I = -\int \frac{1}{u} du = -\ln|\sin x - \cos x|$

$$(28) \int \tan x \sec x dx$$

### Solution

$$I = \int \frac{\sin x}{\cos^2 x} dx$$

Let  $u = \cos x$ , so that  $du = -\sin x dx$ ,

and  $I = -\int u^{-2} du = u^{-1} = \sec x$

$$(29) \int \arcsin\left(\frac{x}{3}\right) dx$$

### Solution

By Parts, integrating 1 and differentiating  $\arcsin\left(\frac{x}{3}\right)$ :

$$I = x \arcsin\left(\frac{x}{3}\right) - \int x \left( \frac{1}{\sqrt{1 - \left(\frac{x}{3}\right)^2}} \right) \left(\frac{1}{3}\right) dx$$

$$= x \arcsin\left(\frac{x}{3}\right) - \int \frac{x}{\sqrt{9 - x^2}} dx$$

Let  $u = x^2$ , so that  $du = 2x dx$

$$\text{Then } I = x \arcsin\left(\frac{x}{3}\right) - \frac{1}{2} \int \frac{1}{\sqrt{9 - u}} du$$



$$\begin{aligned}
&= x \arcsin\left(\frac{x}{3}\right) - \frac{1}{2}(9-u)^{\frac{1}{2}}\left(\frac{1}{\frac{1}{2}}\right)(-1) \\
&= x \arcsin\left(\frac{x}{3}\right) + (9-x^2)^{\frac{1}{2}}
\end{aligned}$$

$$(30)^* \int \frac{\sin x}{\sin x + \cos x} dx$$

### Solution

Note that  $\int \frac{\sin x - \cos x}{\sin x + \cos x} dx = -\ln|\sin x + \cos x|$ ,

as  $\frac{d}{dx}(\sin x + \cos x) = \cos x - \sin x$

Let  $\sin x = A(\sin x + \cos x) + B(\sin x - \cos x)$

Then  $A + B = 1$  &  $A - B = 0$ , so  $A = B = \frac{1}{2}$

$$\begin{aligned}
\text{and } I &= \frac{1}{2} \int \frac{\sin x + \cos x}{\sin x + \cos x} + \frac{\sin x - \cos x}{\sin x + \cos x} dx \\
&= \frac{1}{2}x - \frac{1}{2} \ln|\sin x + \cos x|
\end{aligned}$$

(31) Find a reduction formula for  $I_n = \int_0^1 x^n \sqrt{1-x^2} dx$

### Solution

By Parts, integrating  $x\sqrt{1-x^2}$  and differentiating  $x^{n-1}$ ,

$$\begin{aligned}
I_n &= \left[ -\frac{1}{2} \frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} x^{n-1} \right]_0^1 - \int_0^1 -\frac{1}{2} \frac{(1-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} (n-1)x^{n-2} dx \\
&= -\frac{1}{3} \left[ (1-x^2)^{\frac{3}{2}} x^{n-1} \right]_0^1 + \frac{1}{3} (n-1) \int_0^1 (1-x^2)^{\frac{3}{2}} x^{n-2} dx \\
&= -\frac{1}{3} (0-0) + \frac{1}{3} (n-1) \int_0^1 (1-x^2) \sqrt{1-x^2} x^{n-2} dx
\end{aligned}$$

$$= \frac{1}{3}(n-1)\{I_{n-2} - I_n\}$$

so that  $I_n(3 + (n-1)) = (n-1)I_{n-2}$

and hence  $I_n = \frac{n-1}{n+2}I_{n-2}$

$$(32) \int_1^e (\ln x)^2 dx$$

### Solution

By Parts, integrating 1 and differentiating  $(\ln x)^2$ :

$$\begin{aligned} I &= [x(\ln x)^2]_1^e - \int_1^e x(2\ln x) \left(\frac{1}{x}\right) dx \\ &= e - 2 \int_1^e \ln x dx \end{aligned}$$

Then, by Parts again, integrating 1 and differentiating  $\ln x$ :

$$\begin{aligned} I &= e - 2\{[x\ln x]_1^e - \int_1^e x \left(\frac{1}{x}\right) dx\} \\ &= e - 2e + 2(e - 1) = e - 2 \end{aligned}$$

$$(33) \int 2^x dx$$

### Solution

$$\text{Let } 2 = e^k, \text{ so that } I = \int e^{kx} dx = \frac{1}{k} e^{kx} = \frac{1}{\ln 2} 2^x$$

$$(34) \int \frac{1}{x^2+6x+18} dx$$

**Solution**

$$I = \int \frac{1}{(x+3)^2+9} dx = \frac{1}{3} \arctan\left(\frac{x+3}{3}\right)$$

$$(35) \int \frac{x^2}{1+x^6} dx$$

**Solution**

Let  $u = x^3$ , so that  $du = 3x^2 dx$ ,

$$\text{and } I = \frac{1}{3} \int \frac{1}{1+u^2} du = \frac{1}{3} \arctan u = \frac{1}{3} \arctan(x^3)$$

$$(36) \int \frac{1}{(2x^2+3)^{\frac{3}{2}}} dx$$

**Solution**

Let  $2x^2 = 3 \tan^2 \theta$ , so that  $x = \sqrt{\frac{3}{2}} \tan \theta$  and  $dx = \sqrt{\frac{3}{2}} \sec^2 \theta d\theta$

$$\text{Then } I = \sqrt{\frac{3}{2}} \int \frac{\sec^2 \theta}{3^{\frac{3}{2}} (\tan^2 \theta + 1)^{\frac{3}{2}}} d\theta$$

$$= \frac{1}{3\sqrt{2}} \int \frac{\sec^2 \theta}{\sec^3 \theta} d\theta = \frac{1}{3\sqrt{2}} \int \cos \theta d\theta = \frac{1}{3\sqrt{2}} \sin \theta$$

$$\text{Now } \sin^2 \theta = 1 - \cos^2 \theta = 1 - \frac{1}{1 + \tan^2 \theta} = 1 - \frac{1}{1 + \frac{2}{3}x^2}$$

$$= 1 - \frac{3}{3+2x^2} = \frac{2x^2}{3+2x^2}$$

$$\text{so that } I = \frac{1}{3\sqrt{2}} \sqrt{\frac{2x^2}{3+2x^2}} = \frac{x}{3\sqrt{3+2x^2}}$$

$$(37)** \int \sqrt{4x^2 - 1} dx$$

### Solution

Let  $2x = \sec\theta$ , so that  $2dx = \tan\theta \sec\theta d\theta$

$$\text{Then } I = \int \sqrt{\sec^2\theta - 1} \left(\frac{1}{2}\right) \tan\theta \sec\theta d\theta$$

$$= \frac{1}{2} \int \tan^2\theta \sec\theta d\theta$$

$$= \frac{1}{2} \int (\sec^2\theta - 1) \sec\theta d\theta$$

$$= \frac{1}{2} \int \sec^3\theta d\theta - \frac{1}{2} \int \sec\theta d\theta$$

Then see (4) for  $\int \sec\theta d\theta$  and (17) for  $\int \sec^3\theta d\theta$ .

Alternative method: See Integration Exercises - Part 3 (Hyperbolic Functions).

$$(38) \int \frac{4x+5}{\sqrt{4-6x-x^2}} dx$$

### Solution

$$I = -2 \int \frac{-6-2x}{\sqrt{4-6x-x^2}} dx - \int \frac{7}{\sqrt{4-6x-x^2}} dx$$

$$= -\frac{2(4-6x-x^2)^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} - 7 \int \frac{1}{\sqrt{13-(x+3)^2}} dx$$

$$= -4(4-6x-x^2)^{\frac{1}{2}} - 7 \arcsin\left(\frac{x+3}{\sqrt{13}}\right)$$

$$(39) \int \frac{\sin\sqrt{x}}{\sqrt{x}} dx$$

### Solution

Let  $y = \sqrt{x}$ , so that  $dy = \frac{1}{2}x^{-\frac{1}{2}} dx$

and  $I = \int \sin y \cdot 2 dy = -2\cos\sqrt{x}$

(40) Find a reduction formula for  $I_n = \int_0^2 x^n \sqrt{4-x^2} dx$ , and hence show that  $\int_0^2 x^5 \sqrt{4-x^2} dx = \frac{1024}{105}$

### Solution

By Parts, integrating  $x\sqrt{4-x^2}$  and differentiating  $x^{n-1}$ ,

$$\begin{aligned} I_n &= \left[ -\frac{1}{2} \frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} x^{n-1} \right]_0^2 - \int_0^2 -\frac{1}{2} \frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} (n-1)x^{n-2} dx \\ &= -\frac{1}{3} \left[ (4-x^2)^{\frac{3}{2}} x^{n-1} \right]_0^2 + \frac{1}{3} (n-1) \int_0^2 (4-x^2)^{\frac{3}{2}} x^{n-2} dx \\ &= -\frac{1}{3} (0-0) + \frac{1}{3} (n-1) \int_0^2 (4-x^2)\sqrt{4-x^2} x^{n-2} dx \\ &= \frac{1}{3} (n-1) \{4I_{n-2} - I_n\} \end{aligned}$$

so that  $I_n(3 + (n-1)) = 4(n-1)I_{n-2}$

and hence  $I_n = \frac{4(n-1)}{n+2} I_{n-2}$

Then  $\int_0^2 x^5 \sqrt{4-x^2} dx = I_5 = \frac{4(4)}{7} I_3 = \frac{16(4)(2)}{7 \cdot 5} I_1 = \frac{128}{35} I_1$

and  $I_1 = \int_0^2 x\sqrt{4-x^2} dx = -\frac{1}{2} \int_0^2 -2x\sqrt{4-x^2} dx$

$$= -\frac{1}{2} \left[ \frac{(4-x^2)^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right]_0^2 = -\frac{1}{3} (0-8) = \frac{8}{3},$$

$$\text{so that } \int_0^2 x^5 \sqrt{4-x^2} \, dx = \frac{128}{35} \left(\frac{8}{3}\right) = \frac{1024}{105}$$

(41) Find a reduction formula for  $I_n = \int_0^\pi \cos^n x \, dx$ , and hence show that  $\int_0^\pi \cos^4 x \, dx = \frac{3\pi}{8}$

### Solution

$$I_n = \int_0^\pi \cos x \cdot \cos^{n-1} x \, dx$$

Integrating by Parts (integrating  $\cos x$  and differentiating  $\cos^{n-1} x$ ),

$$I_n = [\sin x \cdot \cos^{n-1} x]_0^\pi - \int_0^\pi \sin x (n-1) \cos^{n-2} (-\sin x) \, dx$$

$$= 0 + (n-1) \int_0^\pi (1 - \cos^2 x) \cos^{n-2} x \, dx$$

$$= (n-1)I_{n-2} - (n-1)I_n,$$

$$\text{so that } nI_n = (n-1)I_{n-2}$$

$$\text{and } I_n = \frac{n-1}{n} I_{n-2}$$

$$\text{Hence } \int_0^\pi \cos^4 x \, dx = \frac{3}{4} I_2 = \frac{3}{4} \cdot \frac{1}{2} I_0 = \frac{3}{8} \int_0^\pi 1 \, dx = \frac{3}{8} [x]_0^\pi = \frac{3}{8} (\pi - 0) = \frac{3\pi}{8}$$

(42) Use the substitution  $t = \tan\left(\frac{x}{2}\right)$  to find  $\int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} \, dx$

### Solution

$$\text{Write } I = \int_0^{\frac{\pi}{2}} \frac{1}{2+\sin x} \, dx, \text{ and let } t = \tan\left(\frac{1}{2}x\right)$$

$$\text{Then } \sin x = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = \frac{2 \tan\left(\frac{x}{2}\right)}{\sec^2\left(\frac{x}{2}\right)} = \frac{2t}{1+t^2}$$

$$\text{and } \frac{dt}{dx} = \sec^2\left(\frac{1}{2}x\right) \cdot \frac{1}{2} = \frac{1}{2} (1 + \tan^2\left(\frac{1}{2}x\right)) = \frac{1}{2} (1 + t^2)$$

$$\text{And so } I = \int_0^1 \frac{1}{2 + \left(\frac{2t}{1+t^2}\right)} \cdot \frac{2}{1+t^2} dt = \int_0^1 \frac{1}{1+t^2+t} dt$$

$$\text{Now } 1 + t^2 + t = \left(t + \frac{1}{2}\right)^2 + \frac{3}{4},$$

$$\text{so that } I = \left[ \frac{1}{\left(\frac{\sqrt{3}}{2}\right)} \arctan \left( \frac{t + \frac{1}{2}}{\left(\frac{\sqrt{3}}{2}\right)} \right) \right]_0^1$$

$$= \frac{2}{\sqrt{3}} \{ \arctan(\sqrt{3}) - \arctan\left(\frac{1}{\sqrt{3}}\right) \}$$

$$= \frac{2}{\sqrt{3}} \left( \frac{\pi}{3} - \frac{\pi}{6} \right) = \frac{\pi}{3\sqrt{3}} \text{ or } \frac{\pi\sqrt{3}}{9}$$