Integration Ideas (STEP) (5 pages; 17/7/23)

Refer to Pure: "Integration Methods" first.

(1) The standard substitution method is to write an integral in the form $\int f(x)h\big(g(x)\big)\,dx$, where $\int f(x)dx = g(x)$, and then the substitution $u = g(x)$ will work, provided that $h(u)$ can be integrated.

In some cases it may be easier to spot a derivative, rather than an integral. For example, $\int secx(secx + tanx)^n dx$

$= \int (secx + tanx)^{n-1}(sec^2x + secxtanx)dx$

$= \frac{1}{n}(secx + tanx)^n\ (+c)$

[Note that the term $(secx + tanx)^{n-1}$ is bound to be the $h\big(g(x)\big)$, so we need to be looking out for $\frac{d}{dx}(secx + tanx)$]

(2) It might be possible to rearrange an integrand into the form $f(x)g'(x) + f'(x)g(x) + h(x)$, where $h(x)$ can be integrated easily, in which case $\int f(x)g'(x) + f'(x)g(x)\,dx = f(x)g(x)$

[from the product rule for differentiation, or integration by parts]

Example: $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx$

$\int 2\sqrt{1+x^3}\,dx = 2x\sqrt{1+x^3} - \int 2x.\frac{\frac{1}{2}(3x^2)}{\sqrt{1+x^3}}\,dx$ (by Parts),

so that $\int 2\sqrt{1+x^3} + \frac{3x^3}{\sqrt{1+x^3}}\,dx = 2x\sqrt{1+x^3} + c$

(3) Questions that can be written in the form "Show that $\int_a^b f(x)dx = g(b) - c$" may be tackled by establishing that $\frac{d}{dx}g(x) = f(x)$ and that $g(a) = c$ (where typically a might equal 0).

(4) To find $\int f(x)dx = g(x)$, it might be the case that $g(x)$ appears in a previous part of a question. Differentiate $g(x)$ to see if this is the case. [See STEP 2016, P2, Q7(iv)]

(5) $u = 1/x$ is a potentially useful substitution

Example: $I = \int \frac{1}{x\sqrt{1-x^2}} dx$

Let $u = 1/x$ so that $du = -1/x^2 dx$ and $dx = -x^2 du$,

so that $I = -\int \frac{ux^2}{\sqrt{1-\frac{1}{u^2}}} du = -\int \frac{u^2x^2}{\sqrt{u^2-1}} du$

$= -\int \frac{1}{\sqrt{u^2-1}} du = -arcoshu = -arcosh(1/x)$

(6) Substitutions in definite integrals

Look for a substitution that reverses the limits (and then take advantage of the fact that $\int_a^b f(x)dx = -\int_b^a f(x)dx$).

(i) $\int_0^\infty f(x)dx$: When $u = \frac{1}{x}$, $\int_0^\infty \rightarrow \int_\infty^0$

(ii) $\int_0^a f(x)dx$: When $u = a - x$, $\int_0^a \rightarrow \int_a^0$

Example (from STEP 2015, P3, Q1)

$I = \int_0^\infty f\left(x + \frac{1}{x}\right) dx$, $J = \int_0^\infty \frac{1}{x^2} f\left(x + \frac{1}{x}\right) dx$

Let $u = \frac{1}{x}$, so that $du = -\frac{1}{x^2}\ dx$

Then $J = \int_\infty^0 f\left(\frac{1}{u} + u\right)(-du) = \int_0^\infty f\left(u + \frac{1}{u}\right) du = I$

[Note that $x + \frac{1}{x} \to \frac{1}{u} + u$]

(7) Inequalities of the form $\int_a^\lambda f(x)dx > g(\lambda)$ can sometimes be proved by rewriting $g(\lambda)$ as $\int_a^\lambda h(x)dx$ (by differentiating $g(x)$ to obtain $h(x)$, if $g(a) = 0$) and then showing that

$\int_a^\lambda f(x) - h(x)\ dx > 0$, by rearranging $f(x) - h(x)$ into an expression that is positive for $a < x < \lambda$

(8) When manipulating an inequality involving an integral, it may be possible to simplify the integrand, as shown in the following example:

$\int_0^\lambda (secxcos\lambda + tanx)^n\ dx < \int_0^\lambda (secxcosx + tanx)^n\ dx$,

as $x < \lambda \Rightarrow cosx > cos\lambda$ (given that $0 < \lambda < \frac{\pi}{2}$),

$= \int_0^\lambda (1 + tanx)^n\ dx$

[See STEP 2021, P3, Q3]

(9) Alternative substitutions

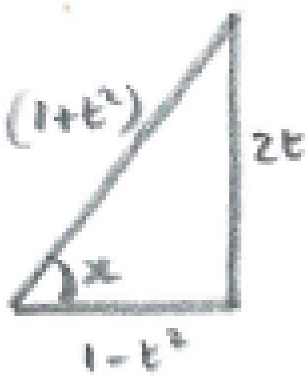
$sec\theta$ can often be used instead of $coshx$, and $tan\theta$ instead of $sinhx$.

(10) $\int \sin(mx)\cos(nx)\,dx = \frac{1}{2}\int \sin(m+n)\,x + \sin(m-n)\,x\,dx$

(11) $t = \tan\left(\frac{x}{2}\right)$ substitution

The substitution $t = \tan\left(\frac{x}{2}\right)$ is usually a method of last resort: it can convert an integrand involving trig. functions to one involving polynomial expressions.

$t = \tan\left(\frac{x}{2}\right) \Rightarrow\ tanx = \frac{2t}{1-t^2}$



Referring to the right-angled triangle shown,

the hypotenuse $= \sqrt{(1-t^2)^2 + 4t^2}$

$= \sqrt{1 + 2t^2 + t^4}\ = 1+t^2$ (conveniently)

$\frac{dt}{dx} = sec^2\left(\frac{x}{2}\right).\frac{1}{2}$, so that $\frac{dx}{dt} = \frac{2}{sec^2\left(\frac{x}{2}\right)} = \frac{2}{1+t^2}$

Example: $\int secx\ dx = \int \frac{1+t^2}{1-t^2}.\frac{2}{1+t^2}\,\text{dt} = 2\int \frac{1}{1-t^2}\,dt$

$= \int \frac{1}{1-t} + \frac{1}{1+t}\ dt = -\ln|1-t| + \ln|1+t| = \ln\left|\frac{1+t}{1-t}\right| = \ln\ \left|\frac{1+2t+t^2}{1-t^2}\right|$

$= \ln\left|\frac{1+t^2}{1-t^2} + \frac{2t}{1-t^2}\right| = \ln|secx + tanx|$

(12) $\int_{-a}^{a} f(-x)\,dx = \int_{-a}^{a} f(x)\,dx$

Proof

Let $u = -x$, so that $du = -dx$, and

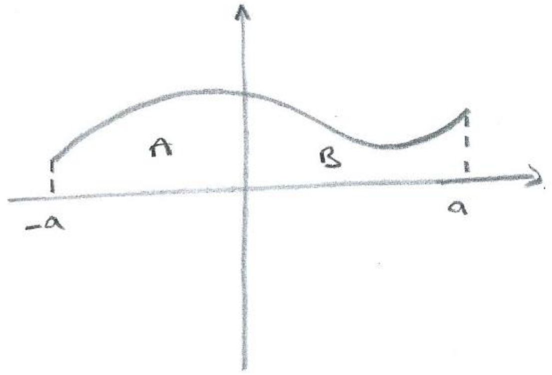
$\int_{-a}^{a} f(-x)\,dx = \int_{a}^{-a} f(u)(-du) = \int_{-a}^{a} f(u)\,du = \int_{-a}^{a} f(x)\,dx$

[Alternatively, considering the integral as an area under a curve, note that $f(-x)$ is the reflection of $f(x)$ about the y-axis, so that

$\int_{-a}^{0} f(-x)\, dx = B = \int_0^a f(x)\, dx$ (referring to the diagram below)

and $\int_0^a f(-x)\, dx = A = \int_{-a}^{0} f(x)\, dx$,

so that $\int_{-a}^{a} f(-x)\, dx = B + A = A + B = \int_{-a}^{a} f(x)\, dx$



(13) $\int_0^a f(\mathrm{a} - x)\, dx = \int_0^a f(x)\, dx$

Proof

Let $u = a - x$, so that $du = -dx$ and

$\int_0^a f(\mathrm{a} - x)\, dx = \int_{\mathrm{a}}^{0} f(\mathrm{u})\, (-\mathrm{du}) = \int_0^a f(u)\, du = \int_0^a f(x)\, dx$

[Note that $f(\mathrm{a} - x)$ is the reflection of $f(x)$ about $x = \frac{a}{2}$.]

(14) To find $\int cosech^2 x\, dx$, note that $\frac{d}{dx}(tanhx) = sech^2 x$ and establish that $\frac{d}{dx}(cothx) = -cosech^2 x$