MAT: Instructive Examples (26 Pages; 2/11/23)

(1) TMUA Specimen P2, Q19

Solution

[A graphical approach looks promising here, but unfortunately turns out to be too complicated.]

The two equations can be rewritten as:

$x^3 + ax^2 - bx - c = 0$ (1)

and $x^3 - ax^2 - bx + c = 0$ (2)

[Given that only the signs of even powers of x differ]

Let $y = -x$

Then (2) becomes $-y^3 - ay^2 + by + c = 0$

or $y^3 + ay^2 - by - c = 0$, which has the same roots as (1).

So, if (1) has one positive and two negative roots, (2) will have one negative and two positive roots.

Answer: B

Comments

Example of rearrangement (substitution)[after observing that only the signs of even powers of x differ]

(2) MAT 2007, Q1(D)

D. The point on the circle

$$(x-5)^2+(y-4)^2=4$$

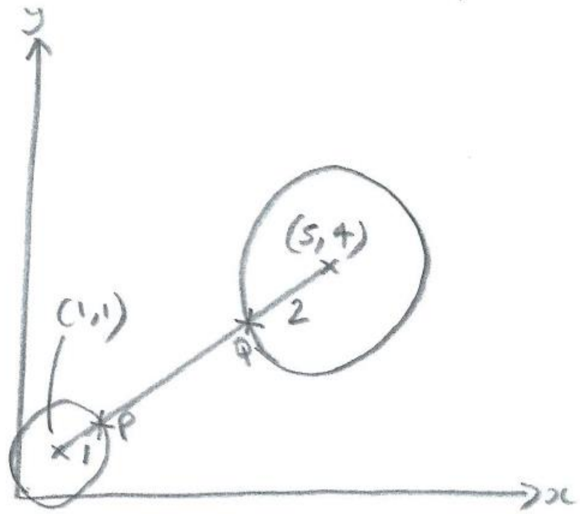
which is closest to the circle

$$(x-1)^2+(y-1)^2=1$$

is

(a) $(3.4, 2.8)$, (b) $(3,4)$, (c) $(5,2)$, (d) $(3.8, 2.4)$.

Solution



The distance between the two centres is 5 (by Pythagoras), and the required point is $\frac{2}{5}$ of the way along the line joining the centres, from the point (5,4).

Taking a weighted average of the two centres [' linear interpolation']:

$\frac{2}{5}(1,1)+\frac{3}{5}(5,4)=(\frac{17}{5},\frac{14}{5})$ or (3.4, 2.8)

So the answer is (a).

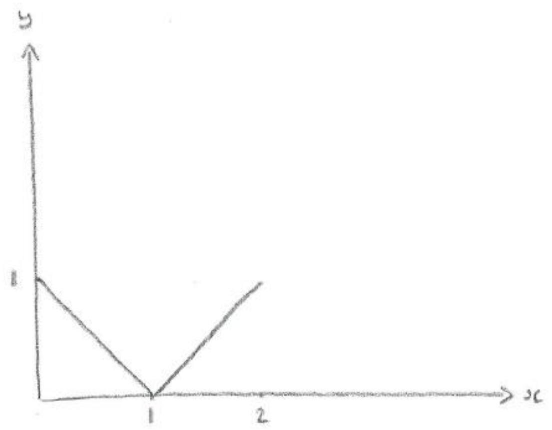
Comments

Simplifying features (3,4,5 triangle) emerges only after diagram has been drawn.

Example of use of linear interpolation.

(3) Sketch the graph of $\sqrt{x^2 - 2x + 1}$ for $0 \leq x \leq 2$

Solution



For $0 \leq x \leq 1, \sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = \sqrt{(1-x)^2} = 1 - x$

For $1 \leq x \leq 2, \sqrt{x^2 - 2x + 1} = \sqrt{(x-1)^2} = x - 1$

Comments

Example of Case by Case approach.

(4) How many solutions are there to $x^3 - 6x^2 + 9x + 2 = 0$?

Solution

$\Leftrightarrow x(x^2 - 6x + 9) = -2$

$\Leftrightarrow x(x-3)^2 = -2$

So one solution, from graph of $y = x(x-3)^2$

Comments

Example of rearrangement, and reformulation of problem (as graphical rather than algebraic problem).

(5) MAT, Specimen Paper B, Q1/G

Solution

The two digit multiples of 13 are 13, 26, 39, 52, 65, 78 & 91

(which doesn't eliminate any of the suggested answers).

Given that the 1st digit is 9, the 2nd digit must be 1; the 3rd digit 3, and the 4th digit 9, so that we have the cycle 913. This accounts for the first 99 digits, so that the last digit must be 9.

ie the answer is (d)

Comments

Look for something that is quick to do (and likely to be worthwhile); ie consider multiples of 13.

“100 digits long”: suggests looking for a pattern

(6) MAT 2009, Q1/D

Solution

[Because of the presence of $(-1)^{n+1}$, it is worth considering separately even and odd n.]

With even n, the LHS becomes $1-2+3-4+\cdots-2m$, writing $n=2m$.

By grouping the terms as $(1-2)+(3-4)+\cdots-2m$, we see that this has a negative value.

So n must be odd, and the LHS becomes

$1-2+3-4+\cdots+(2m+1)$, writing $n=2m+1$

And the terms can be grouped to give

$(1-2)+(3-4)+\cdots([2m-1]-2m)+(2m+1)$

$=m(-1)+(2m+1)=m+1$

So we want $m+1\geq 100$, and hence

$n=2m+1\geq 2(99)+1=199$

So the answer is (c).

Comments

Example of Case by Case.

Experimenting: eliminates even n

(7) Find all positive integer solutions of the equation

$xy - 8x + 6y = 90$

Solution

$xy - 8x + 6y = (x + 6)(y - 8) + 48,$

so that the original equation is equivalent to

$(x + 6)(y - 8) = 42$

The positive integer solutions are given by:

$x + 6 = 7, y - 8 = 6$

$x + 6 = 14, y - 8 = 3$

$x + 6 = 21\ y - 8 = 2$

$x + 6 = 42, y - 8 = 1,$

so that the solutions are:

$x = 1, y = 14$

$x = 8, y = 11$

$x = 15, y = 10$

$x = 36, y = 9$

Comments

- Consider a simpler problem; eg $xy = 90$

- Standard idea for integer-related questions is factorisation.

- Re-read question: “Find all **positive** integer solutions”.

- Consider ALL cases (eg $x + 6 = -7, y - 8 = -6$)

(8) Can n^3 equal $n + 12345670$ (where n is a positive integer)?

Solution

Rearrange to $n^3 - n = 12345670$

$n^3 - n = n(n^2 - 1) = n(n-1)(n+1)$

One of these factors must be a multiple of 3; whereas 12345670 is not a multiple of 3 (since $1 + 2 + 3 + 4 + 5 + 6 + 7 + 0$ isn't a multiple of 3); so answer is No.

Comments

- Large number means that trial and error can probably be ruled out.

- Example of rearrangement.

- Standard idea for integer-related questions is factorisation.

- Useful idea (division by 3) only emerges once you have experimented.

(9) MAT 2007, Q1/J

Solution

Consider $n = 1$ [as this is fairly quick to do]

The LHS is $100 + \frac{1}{2}(100)(101) = 5150$

As the LHS increases with n, the inequality will hold provided that $k < 5150$

So the answer is (d).

Comments

- Experiment with a particular value.

(10) MAT 2008, Q1/B

Solution

Write $L = log_{10}\pi$

[It often helps to have a rough idea of the sizes of the multiple choice options.]

$L \approx \frac{1}{2}$, so that $(b) = \sqrt{2L} \approx 1, (c) = \left(\frac{1}{L}\right)^3 = 8, (d) = \frac{1}{\frac{1}{2}L} \approx 4$

so that we can be fairly sure that the answer is (a), and might just like to check that (a) < (b):

result to prove: $L < \sqrt{2L}$

As $L < 1$ (as $\pi < 10$), $\sqrt{2L} = \sqrt{2}\sqrt{L} > \sqrt{2}\,L > L$, as required.

So the answer is (a).

Comments

Use of approximate values

(11) TMUA Specimen Paper 2, Q19

Solution

[A graphical approach looks promising here, but unfortunately turns out to be too complicated.]

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[Given that only the signs of even powers of x differ]

Let $y = -x$

Then (2) becomes $-y^3 - ay^2 + by + c = 0$

or $y^3 + ay^2 - by - c = 0$, which has the same roots as (1).

So, if (1) has one positive and two negative roots, (2) will have one negative and two positive roots.

Answer: B

(12) MAT 2009, Q1/J

Solution

The presence of $8y^3$ suggests that $(x+2y)^3$ might possibly expand to give the LHS - which it does.

This then gives $x+2y=2^{10}$, and we can simplify matters by writing $x=2u$ (since x has to be even), to give $u+y=2^9$.

Then y can take the values $1, 2, \dots, 2^9-1$ (with $x=2^{10}-2y$), so that there are 2^9-1 such pairs.

So the answer is (c).

Comments

Anything complicated-looking is likely to have a simple interpretation.

(13) MAT 2008, Q1/J

Introduction

This equation can be interpreted as the intersection of the functions $y = (3 + cosx)^2$ and $y = 4 - 2sin^8x$.

In order for the functions to intersect, their ranges must overlap.

Given the relatively complicated nature of these functions, the simplest outcome for this question would be that either the ranges don't overlap at all, or they overlap at one value.

Solution

Note that $(3 + cosx)^2 \geq 4$,

whilst $4 - 2sin^8x \leq 4$

So a sol'n only exists when $cosx = -1$ and $sinx = 0$;

ie at $x = \pi$ (in the range $0 \leq x < 2\pi$)

So the answer is (b).