Inequalities (4 pages; 3/6/23)

Contents

(A) Methods

(1) Convert into an equation, by considering the critical value where equality holds.

(2) If a and b are ≥ 0, then $a > b \Leftrightarrow a^2 > b^2$ (as $y = x^2$ is an increasing function for $x \geq 0$).

eg $|x - 1| > |x + 2| \Leftrightarrow (x - 1)^2 > (x + 2)^2$

(3) If an expression can be arranged into the form $(a - b)^2$, then this will be non-negative.

(4) Use of linear interpolation, to obtain lower or upper bound.

(5) Methods for solving $\frac{x+1}{x-2} < 2x$

Method 1: Multiply both sides by $(x - 2)^2$ (as this is positive, assuming that $x \neq 2$). The resulting cubic will have a factor of

$x - 2$. Consider the regions of the graph.

Method 2: Treat the cases $x - 2 < 0$ and $x - 2 > 0$ separately

Method 3: Rearrange as $\frac{x+1}{x-2} - 2x \ < 0$, and write the LHS as a single fraction. Consider the critical points where either the numerator or the denominator is zero.

Method 4: Sketch $y = \frac{x+1}{x-2}$ and $y = 2x$, and consider the points of intersection.

(B) Notes

(1) Beware of multiplying inequalities by a quantity that is (or could be) negative (eg log(0.5)).

(2) $a < b \Rightarrow \frac{1}{a} + \frac{1}{b} < \frac{1}{a-\delta} + \frac{1}{b+\delta}$ $(\delta > 0)$

(C) Inequalities involving moduli

(1) $|x - 2| > 5$

Method 1

x is more than 5 away from 2, and so has to be either < -3 or > 7

Method 2

$|x - 2| > 5 \Leftrightarrow (x - 2)^2 > 25$ etc

Method 3

Case 1: $x - 2 \geq 0$; Case 2: $x - 2 < 0$

Method 4

Draw graphs of $y = |x - 2|$ and $y = 5$

(2) $2 < |x + 3| < 7$

Method 1 Distance of x from -3 is between 2 and 7

So $-10 < x < -5 \; or -1 < x < 4$

Method 2 $2 < |x + 3| < 7 \Leftrightarrow 4 < (x + 3)^2 < 49$ etc

Method 3 Case 1: $x + 3 \geq 0$; Case 2: $x + 3 < 0$

(3) $|x-2|>|x-5|$

Method 1 x has to be further from 2 than from 5.

It is equidistant when $x=\frac{7}{2}$, and so $x>\frac{7}{2}$.

Method 2 $|x-2|>|x-5| \Leftrightarrow (x-2)^2>(x-5)^2$ etc

Method 3 Case 1: $x<2$; Case 2: $2 \leq x<5$; Case 3: $x \geq 5$

Method 4 Draw the graphs of $y=|x-2|$ and $y=|x-5|$