Improper Integrals (3 pages; 19/9/18)

(1) These are definite integrals of the following types:

(a) There is a limit of ∞ or $-\infty$.

(b) The integrand (the expression being integrated) is undefined at one of the limits (often zero).

(c) The integrand is undefined at a point between the limits.

(2) In each case, the integral may or may not have a value, depending on whether the area under the relevant graph has a finite value.

For (a) and (b), the standard technique is to replace the relevant limit by a variable (such as a), carry out the integration, and then see whether the resulting expression in a tends to a limit as a approaches the original limit (of integration).

For (c), the interval is split into two (or more) sections, and the integrals determined as for (b). Each integral has to be defined (ie have a value), in order for the overall integral to to be defined.

(3) Example A: $\int_{-\infty}^{-1} e^x \, dx = \lim_{a \to -\infty} \int_a^{-1} e^x \, dx$

$= \lim_{a \to -\infty} [e^x]_a^{-1} = \lim_{a \to -\infty} (e^{-1} - e^a) = e^{-1}$

(4) Example B: $\int_1^{\infty} \frac{1}{x} \, dx = \lim_{a \to \infty} \int_1^a \frac{1}{x} \, dx$

$= \lim_{a \to \infty} [lnx]_1^a = \lim_{a \to \infty} (lna - 0)$, which is undefined.

(5) Example C: $\int_0^1 \frac{1}{\sqrt{x}}\, dx = \lim_{a \to 0^+} \int_a^1 \frac{1}{\sqrt{x}}\, dx$

[Note: Strictly speaking, we are taking the limit as a approaches 0 from above; though often just $a \to 0$ is written.]

$$= \lim_{a \to 0^+} \left[\frac{x^{\frac{1}{2}}}{(\frac{1}{2})}\right]_a^1 = \lim_{a \to 0^+} 2(1 - \sqrt{a}) = 2$$

(6) Example D: $\int_{-2}^2 \frac{1}{x^3}\, dx = \lim_{a \to 0^-} \int_{-2}^a \frac{1}{x^3}\, dx + \lim_{b \to 0^+} \int_b^2 \frac{1}{x^3}\, dx$

$$= \lim_{a \to 0^-} \left[\frac{x^{-2}}{-2}\right]_{-2}^a + \lim_{b \to 0^+} [\frac{x^{-2}}{-2}]_b^2$$

$= \lim_{a \to 0^-} \left(-\frac{1}{2}\right)\left(a^{-2} - \frac{1}{4}\right) + \lim_{b \to 0^+} (-\frac{1}{2})(\frac{1}{4} - b^{-2})$, which is undefined, as both $\lim_{a \to 0^-} a^{-2}$ and $\lim_{b \to 0^+} b^{-2}$ are undefined.

[Note: Each limit has to be defined separately. This is emphasised by having separate variables a and b. It would be incorrect to write:

$$\lim_{a \to 0} \left(-\frac{1}{2}\right)\left(a^{-2} - \frac{1}{4}\right) + \lim_{a \to 0} (-\frac{1}{2})(\frac{1}{4} - a^{-2})$$

$$= \lim_{a \to 0} \left(-\frac{1}{2}\right)\left(-\frac{1}{4} + \frac{1}{4} + a^{-2} - a^{-2}\right) = \lim_{a \to 0} (0) = 0\ ]$$

(7) Example E: $\int_{-1}^1 \frac{1}{x}\, dx = \lim_{a \to 0^-} \int_{-1}^a \frac{1}{x}\, dx + \lim_{b \to 0^+} \int_b^1 \frac{1}{x}\, dx$

$$= \lim_{a \to 0^-} [ln|x|]_{-1}^a + \lim_{b \to 0^+} [ln|x|]_b^1$$

$$= \lim_{a \to 0^-} (ln(-a) - ln1) + \lim_{b \to 0^+} (ln\,1 - lnb)$$

which is undefined, as both $\lim_{a \to 0^-} \ln(-a)$ and $\lim_{b \to 0^+} lnb$ are undefined.

[Note: we cannot interpret $\int_{-1}^{1} \frac{1}{x}\, dx$ as being the sum of two infinite areas, of equal size and opposite sign, to give "$-\infty + \infty = 0$" (as $-\infty + \infty$ is undefined).]

(8) Example F: $\int_{-1}^{\infty} \frac{1}{x^5} dx = \lim_{a \to 0^-} \int_{-1}^{a} \frac{1}{x^5} dx + \lim_{\substack{b \to 0^+ \\ c \to \infty}} \int_{b}^{c} \frac{1}{x^5} dx$

$= \lim_{a \to 0^-} \left[\frac{x^{-4}}{-4}\right]_{-1}^{a} + \lim_{\substack{b \to 0^+ \\ c \to \infty}} \left[\frac{x^{-4}}{-4}\right]_{b}^{c}$,

which is undefined, as both $\lim_{a \to 0^-} a^{-4}$ and $\lim_{b \to 0^+} b^{-4}$ are undefined (though $\lim_{c \to \infty} c^{-4}$ is defined).