Hyperbolas - Q4 [Practice/H](26/5/21)

Use matrices to show that the rectangular hyperbola

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Solution

The asymptotes of $x^2 - y^2 = a^2$ are $y = \pm x$, whilst the asymptotes of $xy = c^2$ are the x and y axes.

So consider a rotation of 45° clockwise.

Then the point (x, y) on the hyperbola $xy = c^2$ is transformed to the point (u, v), where

$$\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} cos\,(-45°) & -sin\,(-45°) \\ sin\,(-45°) & cos\,(-45°) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} x + y \\ -x + y \end{pmatrix}$$

Then $u^2 - v^2 = (u - v)(u + v)$

$$= \frac{1}{\sqrt{2}}(2x).\frac{1}{\sqrt{2}}(2y) = 2xy = 2c^2$$

Relabelling gives $x^2 - y^2 = 2c^2$ (and $a^2 = 2c^2$).