Hyperbolas Q3 [11 marks](2/7/21)

Exam Boards

OCR : -

MEI: -

AQA: -

Edx: Further Pure 1 (Year 2)

The chord PQ, where P and Q are points on the rectangular hyperbola $xy = c^2$, has gradient 1. Show that the locus of the point of intersection of the tangents from P and Q is the line

$y = -x$.

Solution

Let P & Q be the points $\left(ct_1, \frac{c}{t_1}\right)$ & $\left(ct_2, \frac{c}{t_2}\right)$, respectively.

As the gradient of PQ is 1, $\frac{\frac{c}{t_2}-\frac{c}{t_1}}{ct_2-ct_1} = 1$,   [1 mark]

so that $\frac{1}{t_2} - \frac{1}{t_1} = t_2 - t_1$

$\Rightarrow \frac{t_1-t_2}{t_1t_2} = t_2 - t_1$

$\Rightarrow t_1t_2 = -1$, as $t_1 \neq t_2$ (P & Q being distinct points) [2 marks]

The equation of the tangent from P is:

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{dy/_{dt}}{dx/_{dt}}|t = t_1$, where $x = ct$ & $y = \frac{c}{t}$

so that $dy/_{dt} = -\frac{c}{t^2}$ & $dx/_{dt} = c$

and the equation of the tangent from P is

$\frac{y-\frac{c}{t_1}}{x-ct_1} = \frac{\left(-\frac{c}{t_1{}^2}\right)}{c} \Rightarrow t_1{}^2y - t_1c = -(x - ct_1)$

$\Rightarrow t_1{}^2y = -x + 2ct_1$ (1) [3 marks]

Similarly, the equation of the tangent from Q is $t_2{}^2y = -x + 2ct_2$

and these lines intersect where

$t_1{}^2y - 2ct_1 = t_2{}^2y - 2ct_2$,   [1 mark]

so that $y(t_1{}^2 - t_2{}^2) = 2c(t_1 - t_2)$

and $y = \frac{2c}{t_1+t_2}$ (as $t_1 \neq t_2$) [1 mark]

Then, from (1), $x = 2ct_1 - \frac{2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1{}^2+2ct_1t_2-2ct_1{}^2}{t_1+t_2}$

$= \frac{2ct_1t_2}{t_1+t_2}$ [2 marks]

and so $\frac{y}{x} = \frac{1}{t_1 t_2} = -1$ (found earlier),

and thus the points of intersection satisfy $y = -x$, as required.

[1 mark]