Hyperbolas - Q2 [10 marks](26/5/21)

Exam Boards

OCR : -

MEI: -

AQA: -

Edx: Further Pure 1 (Year 2)

(i) Given that the tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at the point $(acosht, bsinht)$ (with equation $yasinht = xbcosht - ab$) meets the asymptotes of the hyperbola at the points P & Q, show that the mid-point of P and Q is $(acosht, bsinht)$. [6 marks]

(ii) In the case where $b = a$, find the area of the triangle OPQ (where O is the Origin). [4 marks]

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Solution

(i) The asymptotes of the hyperbola are $y = \pm\frac{b}{a}x$ [1 mark]

The tangent to the hyperbola at $(acosht, bsinht)$ meets the asymptote $y = \frac{b}{a}x$ at P (say), where $bxsinht = xbcosht - ab$

[1 mark]

and the asymptote $y = -\frac{b}{a}x$ at Q where

$-bxsinht = xbcosht - ab$ [1 mark]

so that P is the point $(\frac{a}{cosht-sinht}, \frac{b}{cosht-sinht})$ [1 mark]

and Q is the point $(\frac{a}{cosht+sinht}, \frac{-b}{cosht+sinht})$ [1 mark]

The mid-point of P & Q is therefore

$$(\frac{1}{2}\left[\frac{a}{cosht-sinht}+\frac{a}{cosht+sinht}\right], \frac{1}{2}\left[\frac{b}{cosht-sinht}+\frac{-b}{cosht+sinht}\right])$$

$= \left(\frac{acosht}{cosh^2t-sinh^2t}, \frac{bsinht}{cosh^2t-sinh^2t}\right) = (acosht, bsinht)$, as required.

[1 mark]

(ii) The two asymptotes are at right angles to each other, so that the required area, $A = \frac{1}{2}OP.OQ$ [1 mark]

Then $4A^2 = \left(\left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}} + \left(\frac{a}{cosht-sinht}\right)^{\mathbf{2}}\right)$

$\times\left(\left(\frac{a}{cosht+sinht}\right)^{\mathbf{2}}+\left(\frac{-a}{cosht+sinht}\right)^{\mathbf{2}}\right)$ [1 mark]

$=\left(\frac{2a^2}{(cosht-sinht)^2}\right)\left(\frac{2a^2}{(cosht+sinht)^2}\right)$ [1 mark]

$=\frac{4a^4}{(cosh^2t-sinh^2t)^2}=4a^4$

and therefore $A=a^2$ [1 mark]