Hooke's Law (9 pages; 10/3/21)

(1) Hooke's law can be expressed in 3 ways:

(i) $T = \frac{EAx}{l}$, where E is Young's modulus and A is the cross-sectional area of a string (or wire/spring)

E depends only on the nature of the material, and has S.I. units of Nm^{-2} or Pascals (Pa).

This version is used in Physics, but not in A Level Mechanics.

(ii) $T = \frac{\lambda x}{l}$, where λ is the modulus of elasticity (of the string)

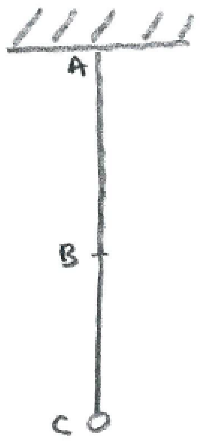
As well as the nature of the material, λ depends on the cross-sectional area of the string. Its S.I. units are N.

This is the version most commonly used in A Level Mechanics.

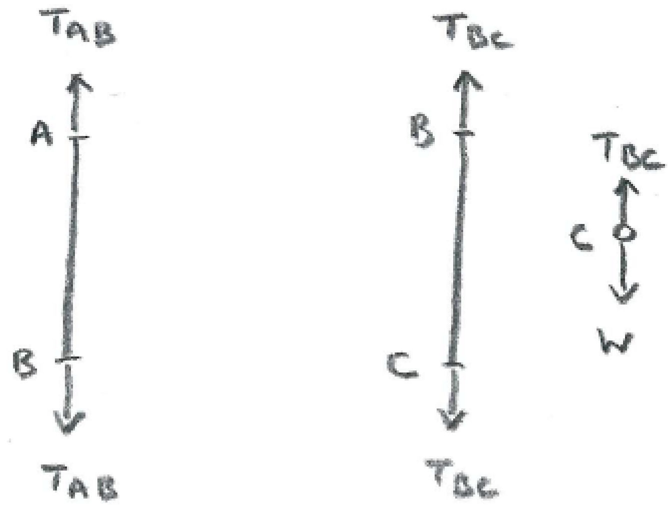
(iii) $T = kx$, where k is the stiffness (of the string)

As well as the nature of the material and the cross-sectional area of the string, k depends on the original length of the string. In other words, k is specific to each piece of string. Its S.I. units are Nm^{-1}.

(2) Consider the situation shown below, where the string AB (with original length l_1, modulus of elasticity λ_1 and extension x_1) is hanging vertically from a fixed point A, and is connected to the string BC (with original length l_2, modulus of elasticity λ_2 and extension x_2), which also hangs vertically and has a load of weight W at C. The strings are assumed to be 'light'; ie having negligible mass.



Force diagrams can be drawn for the string AB, the string BC, and the load at C.



Considering the string AB, the equal forces at the two ends can be justified as follows: Suppose that the forces are T_1 and T_2 (at the top and bottom, respectively). Then, applying N2L to AB,

$T_1 - T_2 = ma$, where m is the mass of the string, and a is its downwards acceleration (in the more general case).

But as m is assumed to be negligible, it is approximately true that $T_1 - T_2 = 0$; ie the two tensions can be assumed to be equal, so that $T_1 = T_2 = T_{AB}$. [Note that, in the case of a spring, it is possible for T_{AB} to be negative; in which case the spring is in compression.]

By Newton's 3rd law, the force on AB at B (ie T_{AB}) will be equal (but with opposite direction) to the force on BC at B (ie T_{BC}), so that $T_{AB} = T_{BC}$.

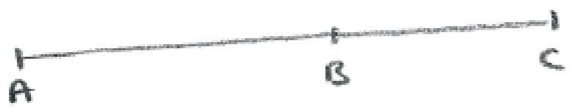
In the force diagram for C, the upward force is T_{BC}, by Newton's 3rd law. As the system is at rest, $W - T_{BC} = 0$, by Newton's 2nd law. Thus $T_{AB} = T_{BC} = W$, and so $\frac{\lambda_1 x_1}{l_1} = \frac{\lambda_2 x_2}{l_2} = W$

Note that we cannot bypass the load at C by saying that the downward force on BC is W, because W is the gravitational force on the load C; not on the string. Instead T_{BC} has to be introduced as the reaction force between the string and the load.

When the system is at rest (or in equilbrium; ie when the acceleration is zero), the effect is the same as if a force of W were applied directly to the string. But if the system has a non-zero acceleration (so that $W \neq T_{BC}$), then this would not be the case.

The string could of course be pulled on by a force F. In that case, F would equal T_{BC}, by the definition of the tension (and this would be true whether the system were accelerating or not).

(3) Consider now the situation where string AB (with original length l_1 and modulus of elasticity λ_1) is horizontal, with A fixed, and joined to the string BC (with original length l_2 and modulus of elasticity λ_2), which is also horizontal, with C fixed. Suppose that the distance AC is $l_1 + l_2 + x$. Let the extensions of AB and BC be x_1 and x_2, so that $x_1 + x_2 = x$.



Then $T_{AB} = T_{BC}$, as before, and $T_{AB} = \frac{\lambda_1 x_1}{l_1}$ and $T_{BC} = \frac{\lambda_2 x_2}{l_2}$

Therefore $\frac{\lambda_1 x_1}{l_1} = \frac{\lambda_2 x_2}{l_2} = \frac{\lambda_2 (x - x_1)}{l_2}$,

so that $x_1(\lambda_1 l_2 + \lambda_2 l_1) = \lambda_2 l_1 x$

and thus $x_1 = \frac{\lambda_2 l_1 x}{\lambda_1 l_2 + \lambda_2 l_1}$

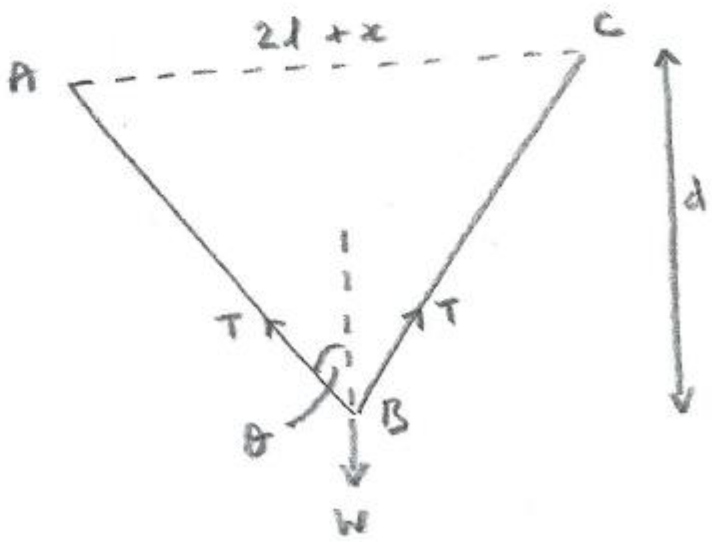
(4) Suppose that a load of weight W is now applied at B, such that, in an equilibrium position, B is at a distance d below its original level (though not necessarily directly below its original position).

Special Case: $l_1 = l_2 = l$ and $\lambda_1 = \lambda_2 = \lambda$, and also $x_1 = x_2 = \frac{1}{2}x$;

ie the strings AB and BC are identical, and B is directly below the mid-point of AC.

By symmetry, the new tensions in the strings are equal; say T.

The force diagram for the load at B is shown below.



Resolving vertically, $2Tcos\theta = W$ (1)

Also, $tan\theta = \frac{l+\frac{x}{2}}{d}$ (2) and, if y is the new extension of AB,

$l + y = \sqrt{(l+\frac{x}{2})^2 + d^2}$ (3) and $T = \frac{\lambda y}{l}$ (4)

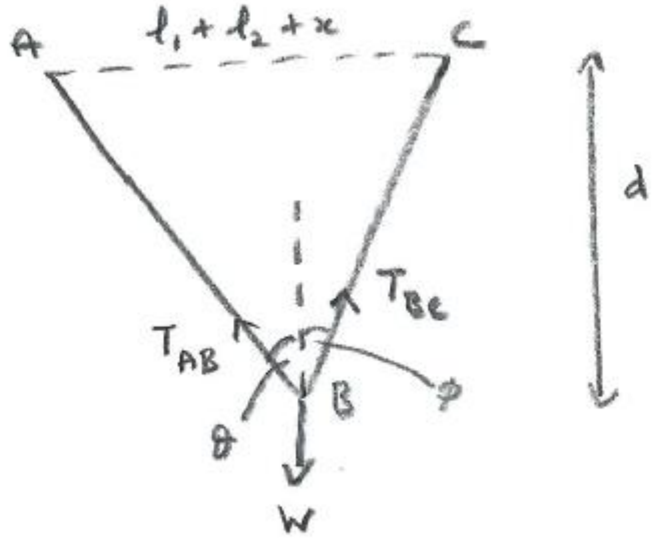
Suppose that $\lambda = 10, l = 2, x = 2, d = 4$. Then we can find W as follows:

From (2), $tan\theta = \frac{3}{4}$, so that $cos\theta = \frac{4}{5}$;

so that (3) $\Rightarrow 2 + y = 5$, and (4) $\Rightarrow T = \frac{30}{2} = 15$

Then, from (1), $W = 30\left(\frac{4}{5}\right) = 24$

General case



Suppose that $l_1 = 1, l_2 = 2, \lambda_1 = 8, \lambda_2 = 12, x = 2, d = 4$.

Let y_1 and y_2 be the new extensions of AB and BC respectively.

Here we can't take advantage of symmetry. Equations can be set up (as shown below), but they would be difficult to solve.

Hence exam questions are likely to avoid this situation, and be based instead on the special case, where the load hangs below the mid-point.

The equations are:

Resolving vertically, $T_{AB}cos\theta + T_{BC}cos\phi = W$ (1)

Resolving horizontally, $T_{AB}sin\theta = T_{BC}sin\phi$ (2)

And $T_{AB} = \frac{\lambda_1 y_1}{l_1} = 8y_1$ (3a) and $T_{BC} = \frac{\lambda_2 y_2}{l_2} = 6y_2$ (3a)

Then $cos\theta = \frac{d}{l_1+y_1} = \frac{4}{1+y_1}$ (4a) and $cos\phi = \frac{d}{l_2+y_2} = \frac{4}{2+y_2}$ (4b)

Also, $dtan\theta + dtan\phi = l_1 + l_2 + x$,

so that $tan\theta + tan\phi = \frac{5}{4}$ (5)

(Allowing for the known values, there are 7 unknowns and 7 equations.)

(5) Multiple springs

(a) Springs in series

Consider two springs of stiffness k_1&k_2 , held in equilibrium in series, as shown in Figure 1.

Figure 1

Draw separate force diagrams for the two springs, as in Figure 2.

Figure 2

By N2L, $T_1 = F$ and $T_2 = F$

(Also, $T_1 = T_2$, by N3L.)

By Hooke's Law, $F = k_1e_1$ & $F = k_2e_2$, where $e_1\&e_2$ are the extensions of the two springs.

Let the stiffness of the combined springs be k.

Then $F = k(e_1 + e_2)$

and so $k = \frac{F}{e_1+e_2} = \frac{F}{\frac{F}{k_1}+\frac{F}{k_2}}$

$\Rightarrow \frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$

Note: This can be extended to more than two springs, by replacing $\frac{1}{k_2}$ with $\frac{1}{k_3} + \frac{1}{k_4}$, and then re-labelling to give $\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2} + \frac{1}{k_3}$

(b) Springs in parallel

Consider two springs of the same original length and stiffnesses $k_1\&k_2$, held in equilibrium in parallel, as shown in Figure 3.

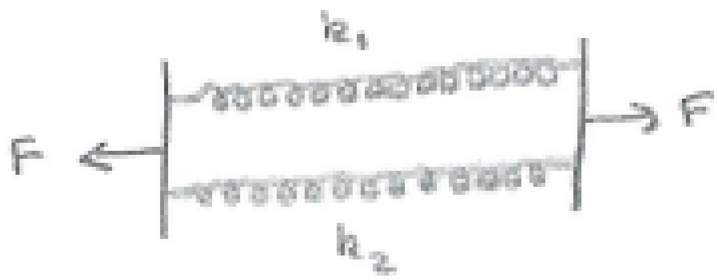


Figure 3

(Note: This system only makes sense if the original lengths are the same, so that when no force is applied, the springs both reach to the two sides.)

The left-hand side of this system is equivalent to that shown in Figure 4, with forces being applied to a light bar (akin to a towbar between a car and a trailer).

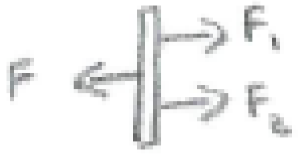


Figure 4

Let the stiffness of the combined springs be k, with extension e.

As the springs have the same original length, their extensions are both equal to e.

Then $F = ke$, $F_1 = k_1 e$, $F_2 = k_2 e$, and $F = F_1 + F_2$

Hence $ke = k_1 e + k_2 e$, and so $k = k_1 + k_2$