Groups – Q4 [Practice/M] $(26/5/21)$

Establish which of the following groups are isomorphic to each other:

(i) $\{0,1,2,3\}$; addition modulo 4

(ii) $\{1,2,4,8\}$; multiplication modulo 15

(iii) $\{3,6,9,12\}$; multiplication modulo 15

(iv) $\{1,3,5,7\}$; multiplication modulo 8

(v) $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\}$; matrix multiplication

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers

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Solution

Groups of order 4 are either cyclic or Klein 4, as established by their Cayley tables (a cyclic group will have elements of period 1,2,4,4; a Klein 4 group will have elements of period 1,2,2,2).

(i) $\{0,1,2,3\}$; addition modulo 4

	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

Cyclic group (0 has period 1, 1 has period 4, 2 has period 2, 3 has period 4).

(ii) {1,2,4,8} ; multiplication modulo 15

	1	2	4	8
1	1	2	4	8
2	2	4	8	1
4	4	8	1	2
8	8	1	2	4

Cyclic group (1 has period 1, 2 has period 4, 4 has period 2, 8 has period 4).

(iii) {3,6,9,12} ; multiplication modulo 15

	3	6	9	12
3	9	3	12	6
6	3	6	9	12
9	12	9	6	3
12	6	12	3	9

Cyclic group (3 has period 4, 6 has period 1, 9 has period 2, 12 has period 4).

(iv) $\{1,3,5,7\}$; multiplication modulo 8

	1	3	5	7
1	1	3	5	7
3	3	1	7	5
5	5	7	1	3
7	7	5	3	1

Klein 4-group (1 has period 1, 3 has period 2, 5 has period 2, 7 has period 2).

(v) $\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\}$; matrix multiplication

[$a = \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}$: 180° rotation

$b = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$: reflection in x-axis

$c = \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$: reflection in y-axis]

	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Klein 4-group (e has period 1, a has period 2, b has period 2, c has period 2).

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers

	1	i	-1	$-i$
1	1	i	-1	$-i$
i	i	-1	$-i$	1
-1	-1	$-i$	1	i
$-i$	$-i$	1	i	-1

Cyclic group (1 has period 1, i has period 4, -1 has period 2, $-i$ has period 4).