

Groups – Q3 [12 marks](25/5/21)

Exam Boards

OCR : Add. Pure (Year 1)

MEI: Extra Pure (Year 2)

AQA: Discrete (Year 2)

Edx: FP2 (Year 1)

For the group $\left\{x, 1 - x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \frac{x}{x-1}\right\}$ under composition of functions, where $x \in \mathbb{R}, x \neq 0, 1$:

(i) Establish whether the group is abelian. [5 marks]

(ii) Find the periods of the elements of the group, and hence identify its proper subgroups. [7 marks]

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Solution

(i) Let $e = x, a = 1-x, b = \frac{1}{x}, c = \frac{1}{1-x}, d = \frac{x-1}{x}, f = \frac{x}{x-1}$

The Cayley table is:

	e	a	b	c	d	f
e	e	a	b	c	d	f
a	a	e	d	f	b	c
b	b	c	e	a	f	d
c	c	b	f	d	e	a
d	d	f	a	e	c	b
f	f	d	c	b	a	e

[3 marks]

For example, $ab = d$ but $ba = c$, so $ab \neq ba$, and hence the group is not abelian. [2 marks]

(ii) $a^2 = e$, so a is of order 2 [1 mark]

$b^2 = e$, so b is of order 2 [1 mark]

$c^2 = d, c^3 = c(c^2) = cd = e$ so c is of order 3 [1 mark]

[note that $c(c^2) = c(c)(c) = (c^2)c$, by associativity, so that

$cd = dc$ (even though the group isn't abelian)]

$d^2 = c, d^3 = d(d^2) = dc = e$ so d is of order 3 [1 mark]

$f^2 = e$, so f is of order 2 [1 mark]

Hence the proper subgroups are: $\{e, a\}, \{e, b\}, \{e, f\}, \{e, c, d\}$

[2 marks]