Groups Overview (25/5/21)

Q1 [8 marks]

Multiplication modulo $\boldsymbol{m}$ (or just mod $\boldsymbol{m}$), denoted by $\times_{\boldsymbol{m}}$, is defined on the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \boldsymbol{m}-\mathbf{1}\}$ by carrying out ordinary multiplication and taking the remainder when the product is divided by $\boldsymbol{m}$. For example, $\mathbf{5} \times_{\mathbf{3}} \mathbf{4} = \mathbf{2}$.

Show that $\times_{\mathbf{5}}$ is a closed and commutative binary operation on the set $\{\mathbf{0}, \mathbf{1}, \mathbf{2}, \dots, \mathbf{4}\}$, and identify the inverse of each element, where it exists.

Q2 [18 marks]

(i) Show that the set {1,4,7,13} forms a group, under multiplication modulo 15. [7 marks]

(ii) Find the generators of the group. [6 marks]

(iii) Establish whether the group is cyclic. [2 marks]

(iv) Identify all the subgroups. [3 marks]

Q3 [12 marks]

For the group $\left\{x, 1-x, \frac{1}{x}, \frac{1}{1-x}, \frac{x-1}{x}, \ \frac{x}{x-1}\right\}$ under composition of functions, where $x \in \mathbb{R}, x \neq 0,1$:

(i) Establish whether the group is abelian. [5 marks]

(ii) Find the periods of the elements of the group, and hence identify its proper subgroups. [7 marks]

Q4 [Practice/M]

Establish which of the following groups are isomorphic to each other:

(i) $\{0,1,2,3\}$; addition modulo 4

(ii) $\{1,2,4,8\}$; multiplication modulo 15

(iii) $\{3,6,9,12\}$; multiplication modulo 15

(iv) $\{1,3,5,7\}$; multiplication modulo 8

(v) $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\right\}$; matrix multiplication

(vi) $\{1, i, -1, -i\}$; multiplication of complex numbers