Graphs – Q8: Transformations [Problem/M](20/6/21)

(i) Suppose that we wish to reflect $y = f(x)$ in the line $x = a$. What combination of transformations could be used to do this?

(ii) Find the equation of the line resulting from the reflection of

$y = 2x + 1$ in the line $x = 1$.

Solution

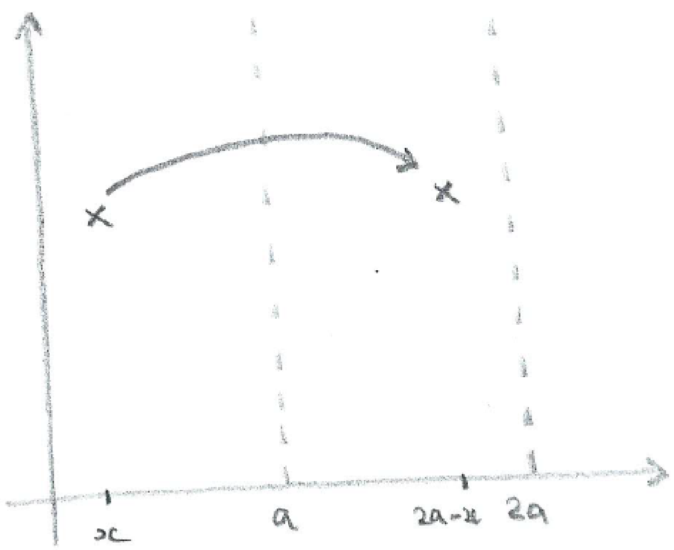
(i) A particular point can be reflected in the line $x = a$ by considering a translation of a to the left, then performing a reflection in the y-axis and translating everything back, by a to the right.

In mathematical terms, x is first of all replaced by $x + a$; then x is replaced by $-x$, and finally x is replaced by $x - a$ (see note below).

Thus $f(x) \to f(x + a) \to f(-x + a) \to f(-[x - a] + a) = f(2a - x)$

[As an aid to memory, consider the reflection of $y = sinx$ about $x = \frac{\pi}{2}$, which is $y = \sin(\pi - x)$]

Alternative approach: $f(2a - x)$ can be justified by observing that when a point is reflected in the line $x = a$, its x coordinate changes from being x to the right of O (in the case where $x > 0$) to being x to the left of $2a$ (as in the example of $y = sinx$). See diagram below.



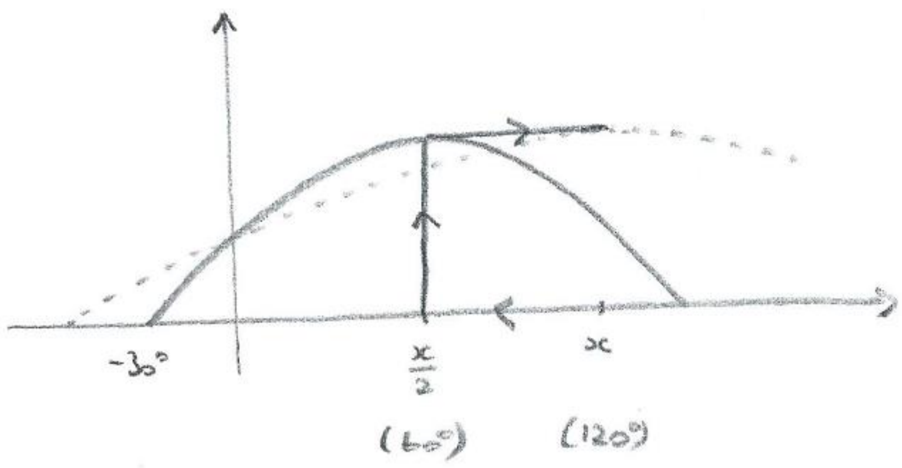
Note: An important point to observe when carrying out composite transformations is that, at any stage of the process, only the following operations are allowed: replacing x with $x + a$ (where a can be negative), or replacing x with kx (where k can be negative).

For example, if we want to stretch $y = \sin(x + 30°)$ by a scale factor 2 in the x-direction, then the point $(x, \sin(x + 30°))$ is moving to $(2x, \sin(x + 30°))$. Making the substitution $u = 2x$, the coordinates of this point on the new curve are

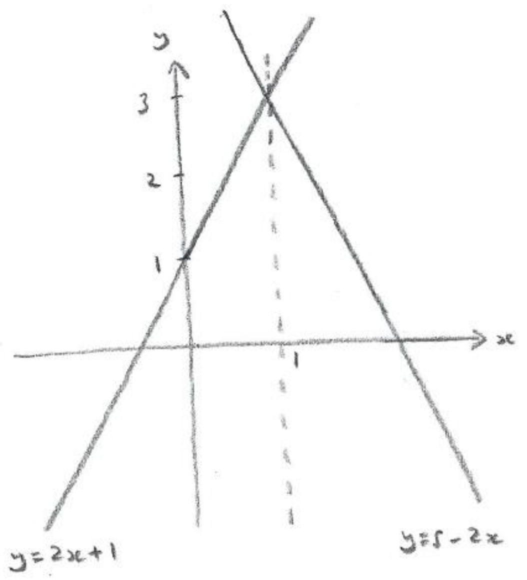
$(u, \sin\left(\frac{u}{2} + 30°\right))$, and re-labelling, to give y as a function of x (rather than u), we have $y = \sin(\frac{x}{2} + 30°)$.

Alternatively (going the other way): the graph of

$y = \sin(\frac{x}{2} + 30°)$ can be obtained as follows: we want the curve with coordinates $(x, \sin(\frac{x}{2} + 30°))$. This can be obtained from the curve with coordinates $(x, \sin(x + 30°))$ by 'looking to the left' of x, to find the point $\left(\frac{x}{2}, \sin\left(\frac{x}{2} + 30°\right)\right)$, and then dragging it back to the right, to give $\left(x, \sin\left(\frac{x}{2} + 30°\right)\right)$ [see diagram below]. (Note that, as this transformation is a stretch, the amount of dragging will depend on the distance from the Origin.) The dragging to the right explains why we see the curve stretching outwards (even though x is being replaced by $\frac{x}{2}$). A similar argument applies in the case of translations (though here the amount of dragging is the same for all points).



(ii) The transformed line is $y = 2(2 - x) + 1 = 5 - 2x$



Check: The transformed line will pass through the point where $y = 2x + 1$ meets the line $x = 1$; ie at $(1, 3)$, and will have a gradient of -2; hence its equation is $\frac{y-3}{x-1} = -2$ etc