Graphs – Q7 [15 marks] (25/5/21)

Exam Boards

OCR : -

MEI: -

AQA: Pure (Year 1)

Edx: -

Sketch the function $y = \frac{x^2}{x-1}$, establishing the location of any local maxima or minima. [15 marks]

Sketch the function $y = \frac{x^2}{x-1}$, establishing the location of any local maxima or minima. [15 marks]

Solution

The curve crosses the x-axis at $x = 0$ (twice), when $y = 0$.

[1 mark]

There is a vertical asymptote at $x = 1$. [1 mark]

$x = 1 + \delta \Rightarrow y = \frac{+}{+} = +$ [1 mark]

[$x = 1 - \delta \Rightarrow y = \frac{+}{-} = -$]

To determine the behaviour of the curve as $x \to \pm\infty$,

$y = \frac{x^2}{x-1} = \frac{x^2-1}{x-1} + \frac{1}{x-1} = x + 1 + \frac{1}{x-1}$

Thus, as $x \to \pm\infty$, $y \to x + 1$ (an 'oblique' asymptote). [2 marks]

[Note that we cannot say that $\lim_{x\to\infty} \frac{x^2}{x-1} = \lim_{x\to\infty} \frac{x}{1-\frac{1}{x}} = \frac{x}{1}$,

as $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \frac{\lim_{x\to\infty} f(x)}{\lim_{x\to\infty} g(x)}$, only when $\lim_{x\to\infty} f(x)$ =constant, and $\lim_{x\to\infty} g(x)$ =constant.]

To see how the curve approaches the oblique asymptote,

consider solutions of $\frac{x^2}{x-1} = x + 1 \Rightarrow x^2 = x^2 - 1$;

ie there are no points of intersection, and so the curve must be approaching the oblique asymptote from below as $x \to -\infty$, and from above as $x \to \infty$. [2 marks]

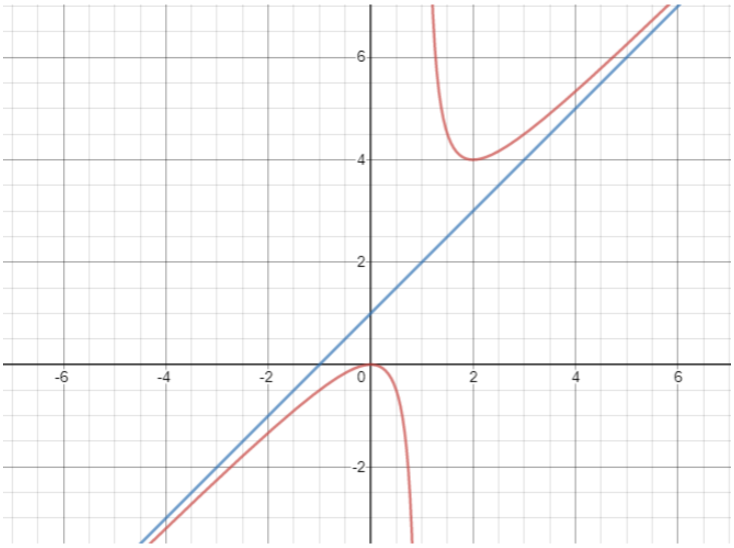
The local maximum of the curve will be at the Origin, where there is the repeated root of $y = 0$. [1 mark]

To find the location of the local minimum, consider solutions of $\frac{x^2}{x-1} = k$; ie $x^2 - kx + k = 0$ [1 mark]

For there to be a solution, the discriminant must be non-negative; ie $(-k)^2 - 4k \geq 0 \Rightarrow k(k-4) \geq 0 \Rightarrow k \leq 0$ or $k \geq 4$ [2 marks]

Thus there are no points of the curve for which $0 < y < 4$, and so the local minimum occurs when $y = 4$ (and $x^2 - 4x + 4 = 0$

$\Rightarrow x = 2$). [2 marks]



[2 marks]