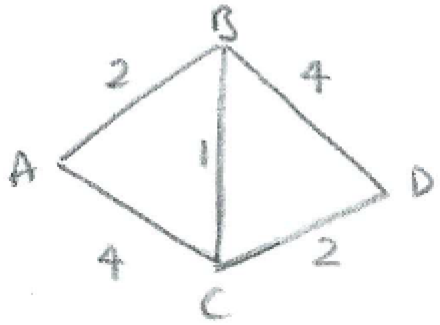
Formulating network problems in LP form

(7 pages; 21/1/20)

(1) Shortest path problem



The aim is to find the shortest path from A to D.

We create binary variables (taking only the values 0 or 1) for each of the arcs AB, BA, AC, CA, BC, CB, BD, DB, CD, DC. Thus, AB = 1 means that we are travelling along AB.

Note that AB, AC, BD & CD are effectively directed arcs (as the shortest path won't involve these arcs being traversed in the opposite direction), so that BA, CA, DB & DC can be set to zero (although, as we will be minimising the path from A to D, the solution would automatically do this anyway).

The objective function (to be minimised) is:

$P = 2AB + 4AC + 4BD + 2CD + BC + CB$

Because we must go along either AB or AC, we have the constraint:

$AB + AC = 1$

(this prevents $AB = 1$ & $AC = 1$ at the same time; as well as $AB = 0$ & $AC = 0$)

and similarly, $BD + CD = 1$

Also, at B the inflow must equal the outflow, so that

$AB + CB = BD + BC$

[This doesn't prevent $AB = 1, CB = 1, BD = 1, BC = 1$, but this possibility will be excluded by the requirement for the total arc length to be minimised.]

and at C, $AC + BC = CB + CD$

Notes

(i) Because this is a minimisation problem, we could in theory just restrict the variables to be non-negative integers; ie the solution would automatically lead to them being binary variables.

(ii) It wouldn't be possible for both $BC = 1$ & $CB = 1$ to occur, as this wouldn't be efficient (as B is being left and then returned to), and so would automatically be excluded.

(iii) Using advanced Simplex methods (Two-stage or Big M), equality constraints such as $AB + AC = 1$ are dealt with by creating a pair of inequalities: $AB + AC \leq 1$ & $AB + AC \geq 1$.

(2) Longest path problem

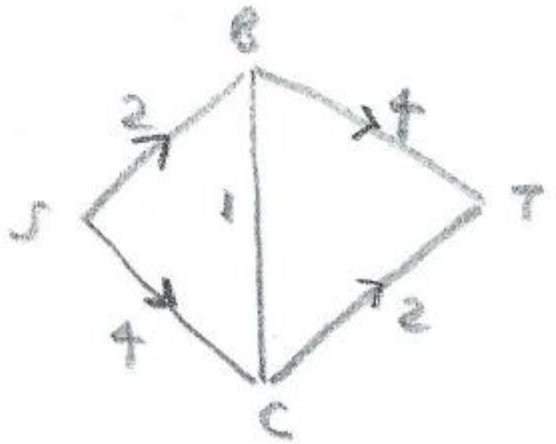
For example, Critical Path Analysis.

As for the shortest path, but maximising P.

However, variables have to be specified as binary now, as values greater than 1 would not be automatically excluded (as we are no longing minimising the total arc length).

[If a program doesn't have the facility to require variables to be binary (but does allow only integer values to be included), then the additional constraints $AB \geq 0, AB \leq 1$ etc will be needed.]

(3) Network Flows



The diagram shows the maximum capacity for each arc. The aim is to maximise the flow across the network.

Networks are often entirely directed. Arcs leading from the source node S and to the sink node T will always be directed.

The variables now represent the flows along the arcs, and are no longer binary. They might not be integers in some cases.

The objective function is: $P = SB + SC$ (to be maximised).

Constraints:

$SB \leq 2, SC \leq 4, BT \leq 4, CT \leq 2, BC \leq 1, CB \leq 1$

At B, inflow equals outflow: $SB + CB = BC + BT$

Similarly for C. (There are no such constraints for S and T.)

Note: The inflow to T will automatically equal the outflow from S.

(4) Matching problem

(i) Suppose that workers A-D are to be matched to tasks 1-4, so that each worker does only one task, and each task is performed by only one worker. In this example, suppose that only the matchings shown in the table are allowable.

	1	2	3	4
A	Y			Y
B	Y		Y	
C		Y	Y	
D			Y	

The aim is to maximise the number of matchings.

(ii) Introduce binary variables (taking only the values 0 or 1) for the allowable matchings: A1, A4, B1, B3, C2, C3, D3

Thus $A1 = 1$ means that worker A is performing task 1.

Maximise $P = A1 + A4 + B1 + B3 + C2 + C3 + D3$,

subject to the following constraints:

rows:

$A1 + A4 \leq 1$ (ie at most one of A1 & A4 can be 1)

$B1 + B3 \leq 1$

$C2 + C3 \leq 1$

[Note: $D3 \leq 1$ isn't needed, as it is a binary variable anyway]

columns:

$A1 + B1 \leq 1$

$B3 + C3 + D3 \leq 1$

(iii) Ideally, P is maximised at 4, so that each of A-D is matched to a task. This is referred to as a 'maximal matching'.

(5) Allocation problem

(i) This is similar to the matching problem, except that each 'association' between the workers and tasks (eg A1) has a 'cost' (which could be a time) attached to it. The optimal allocation is achieved by minimising the total cost. Thus the table below contains the costs of workers being allocated particular tasks.

	1	2	3	4
A	5	2	3	6
B	1	7	2	4
C	5	8	3	1
D	4	4	2	6

As for the matching problem, $A1 = 1$ means that worker A is performing task 1 (so $A1$ etc are binary variables).

If a particular worker cannot carry out a certain task, then this is incorporated by including a very high cost for the relevant cell.

[Note: A possible area for confusion is that A1 represents a binary variable, but the content of what could be described as cell A1 is the cost associated with that combination of worker and task.]

(ii) Minimise $P = 5A1 + 2A2 + \cdots + 6D4$,

[Note that P is minimised this time.]

subject to the following constraints:

$A1 + A2 + A3 + A4 = 1$

$B1 + B2 + B3 + B4 = 1$ etc

$A1 + B1 + C1 + D1 = 1$

$A2 + B2 + C2 + D2 = 1$ etc

(Note that, whereas for Matchings it may not be possible to allocate all workers to a task, for Allocation problems each worker will definitely be allocated to a task.)

(6) Transportation problem

(i) This is similar to the Allocation problem. Now, however, A1 etc represent the numbers of items being transported (eg from warehouse A to shop 1), and so are no longer binary variables (though they must be integers).

All the items in the problem are assumed to be identical. The costs per item are included in the main body of the table. (Note that it is being assumed that this is a good model for arriving at the total cost.)

Now, in addition, there is a supply column and a demand row: the total numbers of items available at warehouses A, B, ... are indicated in the supply column, and the total numbers of items required at shops 1, 2, ... are indicated in the demand row. The total of the supply column will always equal the total of the demand row.

[Note that, whereas the main body of the table contains costs per item, the supply column and demand row contain numbers of items.]

	demand	5	7	4	4
supply		1	2	3	4
3	A	5	2	3	6
6	B	1	7	2	4
9	C	5	8	3	1
2	D	4	4	2	6

(ii) Minimise $P = 5A1 + 2A2 + \cdots + 6D4$,

subject to the following constraints:

$A1 + A2 + A3 + A4 = 3$

$B1 + B2 + B3 + B4 = 6$ etc

$A1 + B1 + C1 + D1 = 5$

$A2 + B2 + C2 + D2 = 7$ etc

(Note that all of the supplies are to be used up, and all of the demands are to be satisfied. Also, each warehouse may deliver to more than one shop.)

Note: $A1 + A2 + A3 + A4 = 3$ can be dealt with using advanced techniques of the Simplex method, by writing it as a pair of inequalities: $A1 + A2 + A3 + A4 \leq 3$ and $A1 + A2 + A3 + A4 \geq 3$ (the ordinary Simplex method will not be able to cope with both types of inequality, but the 'Big M' or 2-Stage methods can be used for this).