First Order Differential Equations (3 pages; 23/2/19)

(1) A differential equation is first order if it doesn't involve any derivatives higher than the first.

(2) The equation $\frac{dy}{dx} = f(x)$ wouldn't usually be described as a differential equation, as it can easily be solved (in principle) by integrating $f(x)$.

Separation of Variables

(3) In general, the equation $\frac{dy}{dx} = f(x)g(y)$ is solved by rearranging (informally) as follows: $\frac{dy}{g(y)} = f(x)dx$, which then gives $\int \frac{1}{g(y)} dy = \int f(x)dx$ [a more rigorous proof exists, but at A Level the result can be assumed]

(4) Example: $\frac{dP}{dt} = kP \ (k > 0)$

$\frac{dP}{P} = kdt$, so that $\int \frac{1}{P} dP = k \int dt,$

and hence $lnP = kt + C,$

giving $P = e^{(kt+C)}$ or Ae^{kt} , where $A = e^C$

Then, if $P = P_0$ when $t = 0, A = P_0,$ so that $P = P_0 e^{kt}.$

(5) Note that $\frac{dy}{dx} = x + y$, for example, cannot be dealt with by the method of Separation of Variables. (An integrating factor can

be used however, as described below. See also "Differential Equations - Substitutions".)

Integrating Factors

(6) A differential equation is referred to as 'linear' if the dependent variable (often y) only appears to the 1st power.

(7) A first order linear equation can be written in the form

$\frac{dy}{dx} + P(x)y = Q(x)$ (A)

Multiplication by an integrating factor, $R(x)$ enables the left-hand side to be written in the form

$\frac{d}{dx}(R(x)y) = R(x)\frac{dy}{dx} + \frac{dR(x)}{dx}.y$

We therefore require $\frac{dR(x)}{dx} = R(x)P(x)$,

so that $\int \frac{1}{R(x)} dR(x) = \int P(x)dx$

and hence $lnR(x) = \int P(x)dx$

and $R(x) = \exp\{\int P(x)dx\}$

(8) On multiplying by $R(x)$, (A) becomes

$\frac{d}{dx}(R(x)y) = R(x)Q(x)$,

so that $R(x)y = \int R(x)Q(x)dx$

(9) In some cases, the left-hand side can be easily rearranged into the form $R(x)\frac{dy}{dx} + \frac{dR(x)}{dx}.y$

Example 1: $sinx\frac{dy}{dx}+secx.y$

Dividing by $cosx$ gives $tanx\frac{dy}{dx}+sec^2x.y=\frac{d}{dx}(tanx.y)$

Example 2: $\frac{dy}{dx}+tanx.y$

Multiplying by $secx$ gives $secx\frac{dy}{dx}+secxtanx.y=\frac{d}{dx}(secx.y)$

Example 3: $tanx\frac{dy}{dx}+y$

Multiplying by $cosx$ gives $sinx\frac{dy}{dx}+cosx.y=\frac{d}{dx}(sinx.y)$

Example 4: $cosx\frac{dy}{dx}-cosecx.y$

Dividing by $sinx$ gives $cotx\frac{dy}{dx}-cosec^2x.y=\frac{d}{dx}(cotx.y)$

(10) Note the standard pattern $\frac{dy}{dx}+\frac{k}{x}y$ $(k\in\mathbb{R})$ for the left-hand side, where the integrating factor will be x^k