Ellipses Q2 [11 marks] (23/5/21)

Exam Boards

OCR : -

MEI: -

AQA: -

Edx: Further Pure 1 (Year 2)

Given the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and circle $x^2+y^2=a^2$, let l_1 be the tangent to the ellipse at the point $(acos\theta, bsin\theta)$ and l_2 be the tangent to the circle at the point $(acos\theta, asin\theta)$. Find the locus of the point of intersection of l_1 & l_2, as θ varies.

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Solution

The equation of l_1 is $\frac{y-bsin\theta}{x-acos\theta}=\frac{dy/d\theta}{dx/d\theta}=\frac{bcos\theta}{-asin\theta}$ (1) [2 marks]

The equation of l_2 is $\frac{y-asin\theta}{x-acos\theta}=\frac{dy/d\theta}{dx/d\theta}=\frac{acos\theta}{-asin\theta}$ (2) [2 marks]

At the intersection of l_1 & l_2,

$x-acos\theta=\frac{-asin\theta}{bcos\theta}(y-bsin\theta)$ from (1)

and $x-acos\theta=\frac{-sin\theta}{cos\theta}(y-asin\theta)$ from (2),

so that $\left(\frac{a}{b}\right)(y-bsin\theta)=y-asin\theta$ [2 marks]

$\Rightarrow ay-absin\theta=by-absin\theta$

$\Rightarrow y=0,$ as $a\neq b$ (otherwise the ellipse would be a circle)

[1 mark]

Then, from (2), $x-acos\theta=\frac{asin^2\theta}{cos\theta}$,

so that $xcos\theta=acos^2\theta+asin^2\theta=a$, and thus $x=\frac{a}{cos\theta}$

[2 marks]

As $-1<cos\theta<1$, x can take values in the range $(-\infty,-a]$ & $[a,\infty)$

Thus the required locus is the set of points on the x-axis in the above range. [2 marks]