Discrete Random Variables (8 pages; 4/2/23)

Contents

(1) Types of Discrete Random variable

(a) defined by a formula; eg Binomial

(b) no formula is readily available, but probabilities can be calculated theoretically; eg sum of values on two dice

(c) probabilities for $X = 1, X = 2, \ldots$ are specified individually; possibly based on experimental data

(2) Variance

$Var(X) = E[(X-\mu)^2]$

$= E[X^2 - 2X\mu + \mu^2]$

$= \sum_x (x^2 - 2x\mu + \mu^2)P(X = x)$

$= [\sum_x x^2 P(X = x)] - [2\mu \sum_x xP(X = x)] + \mu^2 \sum_x P(X = x)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2$

(3) Unbiased estimator for the population variance

The random variable $S^2 = \frac{1}{n-1}([\sum X^2] - n\overline{X}^2)$ is an unbiased estimator for the population variance σ^2.

Proof

$E(S^2) = \frac{1}{n-1}\{[\sum E(X^2)] - nE(\overline{X}^2)\}$

$= \frac{1}{n-1}\{nE(X^2) - nE\left(\overline{X}^2\right)\}$

$= \frac{n}{n-1}\{E(X^2) - E\left(\overline{X}^2\right)\}$

Now $\sigma^2 = E(X^2) - \mu^2$, so that $E(X^2) = \sigma^2 + \mu^2$

Also, $Var(\overline{X}) = E\left(\overline{X}^2\right) - \mu^2$,

and $Var(\overline{X}) = Var(\frac{X_1 + \cdots + X_n)}{n} = \frac{1}{n^2} Var(X_1 + \cdots + X_n)$

$= \frac{1}{n^2}\left(nVar(X_i)\right) = \frac{\sigma^2}{n}$,

so that $E\left(\overline{X}^2\right) = \frac{\sigma^2}{n} + \mu^2$

Then $E(S^2) = \frac{n}{n-1}([\sigma^2 + \mu^2] - \left[\frac{\sigma^2}{n} + \mu^2\right])$

$= \frac{n\sigma^2}{n-1}(1 - \frac{1}{n})$

$= \frac{n\sigma^2}{n-1}(\frac{n-1}{n})$

$= \sigma^2$

(4) $Var(aX + b) = a^2 VarX$

Proof

$Var(aX + b) = E[(aX + b)^2] - [E(aX + b)]^2$

$= E[a^2X^2 + 2abX + b^2] - [aE(X) + b]^2$

$= a^2E(X^2) + 2abE(X) + b^2 - [a^2[E(X)]^2 + 2abE(X) + b^2]$

$= a^2E(X^2) - a^2[E(X)]^2$

$= a^2 VarX$

(5) $Var(X_1 + X_2)$ and $Var(2X_1)$

If $Var(X_1) = Var(X_2)$, and $Y = X_1 + X_2$,

then $Var(Y) = Var(X_1) + Var(X_2) = 2Var(X_1)$, provided that X_1& X_2 are independent

But if $Z = 2X_1$, then $Var(Z) = 4Var(X_1)$.

(6) Covariance

Where X & Y are not necessarily independent:

$Var(aX \pm bY) = a^2 VarX + b^2 VarY \pm 2abCov(X,Y)$,

where $Cov(X,Y) = E(XY) - E(X)E(Y)$

(7) Device for determining $Var(X)$

$E(X^2) = E[X(X-1)] + E(X)$ [as $r(r-1)$ divides into $r!$]

(8) Conditional expectation, $E(X|Y)$

For any random variables X & Y:

First of all consider $f(y) = E(X|Y=y) = \sum_x xP(X=x|Y=y)$

Then, considering y as a random variable, $E(X|Y) = f(Y)$.

Example 1 ("Randomly stopped sum"): $X = X_1 + X_2 + \cdots + X_N$, where X_i is independent of X_j for $i \neq j$, X_i is independent of N, and $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ for each i, and where N is equally likely to be each of the integers 1 to n. [Note though that the X_i don't need to have the same distribution.]

Here $Y = N$, and $E(X|N) = E(X_1 + X_2 + \cdots + X_N) = N\mu$

(9) $E\big(g(X)\big) = E_Y(E(g(X)|Y))$

(for any function g)

Proof

RHS $= E_Y\{\sum_x g(x)P(X = x|Y)\}$

$= \sum_y [\sum_x g(x)P(X = x|Y = y)]P(Y = y)$

$= \sum_y \ \sum_x g(x)P(X = x|Y = y)P(Y = y)$

$= \sum_x \ g(x) \sum_y \ P(X = x|Y = y)(Y = y)$

$= \sum_x \ g(x)P(X = x)$

$= E\big(g(X)\big)$

(10) Law of total expectation

For any random variables X & Y, $E(X) = E_Y\{E(X|Y)\}$ (*)

where $E_Y(E(X|Y)) = \sum_y E(X|Y = y)P(Y = y)$

$= \sum_y f(y)P(Y = y)$

[(*) applies to continuous random variables as well]

Proof: Set $g(X) = X$ in (2).

Example 1 (again): , $E(X) = E_N\{E(X|N)\} = E_N\{N\mu\} = \mu E_N\{N\}$

And $E_N\{N\}[\equiv E(N)] = \sum_{r=1}^{n} r \,.\, \frac{1}{n} = \frac{1}{n} \,.\frac{1}{2}\, n(n + 1) = \frac{1}{2}(n + 1)$

So $E(X) = \frac{\mu}{2}(n + 1)$

(11) Law of total variance

For any random variables X & Y,

$Var(X) = E_Y(Var(X|Y)) + Var_Y(E(X|Y))$ (**)

[(**) applies to continuous random variables as well]

Proof

RHS $= E_Y\{E(X^2|Y) - [E(X|Y)]^2\}$

$+E_Y\{[E(X|Y)]^2\} - \{E_Y[E(X|Y)]\}^2$

$= E(X^2) - E_Y\{[E(X|Y)]^2\} + E_Y\{[E(X|Y)]^2\} - [E(X)]^2$

$= E(X^2) - [E(X)]^2 = Var(X)$

[See "PGF – Exercises" for a (longer) proof using probability generating functions.]

Example 1 (again): $Var(X) = E_N(Var(X|N)) + Var_N(E(X|N))$

Now, $Var(X|N) = Var(X_1 + X_2 + \cdots + X_N) = N\sigma^2$,

and so $E_N(Var(X|N) = E_N(N\sigma^2) = \sigma^2.\frac{1}{2}(n+1)$

And $Var_N\big(E(X|N)\big) = Var_N(N\mu) = \mu^2 Var_N(N)$,

and $Var_N(N) = E(N^2) - [E(N)]^2$

$= \{\sum_{r=1}^{n} r^2 \,. \frac{1}{n}\} - [\frac{1}{2}(n+1)]^2$

$= \frac{1}{n} \,. \frac{1}{6}n(n+1)(2n+1) - \frac{1}{4}(n+1)^2$

$= \frac{(n+1)}{12}[2(2n+1) - 3(n+1)]$

$= \frac{(n+1)(n-1)}{12}$

So $Var(X) = \frac{\sigma^2}{2}(n+1) + \frac{\mu^2(n+1)(n-1)}{12}$

Example 2: As Example 1, but N has a more general distribution.

$Var(X) = E_N(N\sigma^2) + \mu^2 Var_N(N) = \sigma^2 E(N) + \mu^2 Var(N)$

$[E(N) \equiv E_N(N)$ & $Var(N) \equiv Var_N(N)]$

Where $N \sim Po(\lambda)$, for example,

$E(X) = \lambda\mu$ and $Var(X) = \lambda(\sigma^2 + \mu^2)$.

Appendices

(A) Discrete Distributions

Uniform: $X \sim$ discrete $U(a, b)$	(i) $P(X = r) = \frac{1}{b-a+1}$ (ii) $E(X) = \frac{1}{2}(n+1)$ (iii) $Var(X) = \frac{1}{12}(n^2 - 1)$
Binomial: $X \sim B(n, p)$	pgf $G_X(s) = (q + ps)^n$
Geometric: $X \sim Geo(p)$ [X is no. of attempts needed for 1st success]	(i) $P(X = r) = q^{r-1}p$ (ii) $P(X \leq k) = 1 - q^k$

	(iii) $E(X) = \frac{1}{p}$ (iv) $Var(X) = \frac{q}{p^2}$ (v) pgf $G_X(s) = \frac{ps}{1-qs}$
Negative Binomial [X is no. of attempts needed for n successes] [Becomes Geometric when $n = 1$]	(i) prob. of nth success on rth attempt: $p_k =$ $\binom{r-1}{n-1} p^{n-1} q^{(r-1)-(n-1)} p$ $= \binom{r-1}{n-1} p^n q^{k-n}$ (ii) $E(X) = \frac{n}{p}$ (iii) $Var(X) = \frac{nq}{p^2}$ (iv) pgf $G_X(s) = \left(\frac{ps}{1-qs}\right)^n$
Poisson: $X \sim Po(\lambda)$	(i) $p_k = \frac{e^{-\lambda}\lambda^k}{k!}$ (ii) pgf $G_X(s) = e^{\lambda(s-1)}$

(B) Results using Probability Generating Functions

If $X_1, X_2, \ldots$ & N are independent random variables, where the X_i have pgf $G_X(s)$, then

(i) $S_N = X_1 + X_2 + \cdots + X_n$ has pgf $G_{S_N}(s) = G_N\big(G_X(s)\big)$

(ii) $E(S_N) = E(N)E(X)$

(iii) $Var(S_N) = E(N)Var(X) + Var(N)[E(X)]^2$