Determinants & Inverses (14 pages; 3/6/19)

Contents

(A) Determinants

(1) Miscellaneous

(i) $|AB| = |A||B|$ (see (ix) for proof)

(ii) $|M||M^{-1}| = |MM^{-1}| = |I| = 1$, so that $|M^{-1}| = \frac{1}{|M|}$

(iii) $|M^T| = |M|$

(iv) Adding a multiple of one column/row to another column/row leaves the determinant unchanged.

For a 3×3 matrix M, with columns $\underline{a}, \underline{b} \& \underline{c}$, $|M| = \underline{a}.(\underline{b} \times \underline{c})$

and $\left|\underline{a} + k\,\underline{b}, \underline{b}, \underline{c}\right| = (\underline{a} + k\underline{b}).(\underline{b} \times \underline{c})$

$= \underline{a}.(\underline{b} \times \underline{c}) + (k\underline{b}).(\underline{b} \times \underline{c}) = \underline{a}.(\underline{b} \times \underline{c})$,

as $\underline{b} \times \underline{c}$ is perpendicular to $\underline{b}$

(v) Swapping 2 columns/rows changes the sign

(vi) If 2 rows/columns are identical, then the determinant is zero [follows from (v)]

(vii) $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 & a_1 \\ b_2 & c_2 & a_2 \\ b_3 & c_3 & a_3 \end{vmatrix} = \begin{vmatrix} c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \\ c_3 & a_3 & b_3 \end{vmatrix}$

ie the determinant is unchanged if the columns (or rows) are interchanged cyclicly [follows from (v)]

(viii) $\left| k\underline{a}, \underline{b}, \underline{c} \right| = k \left| \underline{a}, \underline{b}, \underline{c} \right|$

as $(k\underline{a}).(\underline{b} \times \underline{c}) = k[\underline{a}.(\underline{b} \times \underline{c})]$

(ix) The determinant of a 3×3 matrix is the volume scale factor of the associated transformation. This provides a proof for

$|AB| = |A||B|$, when successive transformations are applied.

(2) Manipulating determinants

[The entries of the determinant may be numbers or letters.]

(i) Gaussian elimination

[This is a standard procedure, which always works, but usually involves awkward fractions. It is also used (using matrices, rather than determinants) to solve simultaneous equations. Generally though, it is simpler to work out the determinant by the usual expansion.]

The aim is to obtain a triangular form, such as $\begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{vmatrix}$

[When solving simultaneous equations which have been reduced to a triangular form, such as

$$\begin{pmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 6 \\ 7 \\ 8 \end{pmatrix}, z \text{ is obtained from the 3rd row;}$$

then y from the 2nd, and x from the 1st.]

Example: $\begin{vmatrix} 2 & 4 & 7 \\ 3 & 5 & 8 \\ 4 & 7 & 9 \end{vmatrix}$

Step 1: $R2 \rightarrow R2 - \frac{3}{2}R1$, to give $\begin{vmatrix} 2 & 4 & 7 \\ 0 & -1 & -\frac{5}{2} \\ 4 & 7 & 9 \end{vmatrix}$

or $-\frac{1}{2} \begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 4 & 7 & 9 \end{vmatrix}$

Step 2: $R3 \rightarrow R3 - \frac{4}{2}R1$, to give $-\frac{1}{2} \begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & -1 & -5 \end{vmatrix}$

Step 3: $R3 \rightarrow R3 - \frac{(-1)}{2}R2$, to give $-\frac{1}{2} \begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & -\frac{5}{2} \end{vmatrix}$

or $\frac{1}{4} \begin{vmatrix} 2 & 4 & 7 \\ 0 & 2 & 5 \\ 0 & 0 & 5 \end{vmatrix} = \frac{1}{4}(2)(2)(5) = 5$

(ii) Alternative strategy

In many cases it is possible to take advantage of some of the following devices. These are illustrated in the examples below.

(a) Take out any common factors of a row or column.

(b) If an entry is repeated in a row or column, subtract the appropriate column or row, to produce a zero entry.

(c) Once a row such as $(1, 0, 0)$ has been obtained, multiples of this can be subtracted from other rows to produce more zeros.

(iii) Examples

Example 1

Factorise $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Solution

Replacing R1 with R1 - R2 (where R1 means row 1),

$$\Delta = \begin{vmatrix} 0 & x - y & x^2 - y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

This has the advantage of creating a 0 in R1 and obtaining a common factor for the other elements of the row.

Thus $\Delta = (x - y) \begin{vmatrix} 0 & 1 & x + y \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$

Similarly, we can replace R2 with R2 - R3, to give

$$(x - y) \begin{vmatrix} 0 & 1 & x + y \\ 0 & y - z & y^2 - z^2 \\ 1 & z & z^2 \end{vmatrix}$$

so that $\Delta = (x - y)(y - z) \begin{vmatrix} 0 & 1 & x + y \\ 0 & 1 & y + z \\ 1 & z & z^2 \end{vmatrix}$

$= (x - y)(y - z)(y + z - x - y) = (x - y)(y - z)(z - x)$

Example 2

Show that $\begin{vmatrix} -2 & 1 & 2k \\ -1 & 1 & k+1 \\ 2 & k-1 & 1 \end{vmatrix}$ is independent of k

Solution

(The strategy used here is to create zeros wherever possible.)

$$R1 \to R1 - R2 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ -1 & 1 & k+1 \\ 2 & k-1 & 1 \end{vmatrix}$$

$$R2 \to R2 - R1 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ 0 & 1 & 2 \\ 2 & k-1 & 1 \end{vmatrix}$$

$$R3 \to R3 + 2 \times R1 \Rightarrow \begin{vmatrix} -1 & 0 & k-1 \\ 0 & 1 & 2 \\ 0 & k-1 & 2k-1 \end{vmatrix}$$

$$= -1(2k - 1 - 2k + 2) = -1$$

Example 3

Evaluate $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ by row and column operations

Solution

Method 1

$$R2 \to R2 - 4 \times R1 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{vmatrix}$$

[Note: Had the 1 not been present, we could have created it by taking out the appropriate factor from the 1st row or column.]

$$R3 \rightarrow R3 - 7 \times R1 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{vmatrix}$$

$$R3 \rightarrow R3 - 2 \times R2 \Rightarrow \begin{vmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{vmatrix}$$

$$= 0$$

Method 2

$$C2 \rightarrow C2 - C1 \Rightarrow \begin{vmatrix} 1 & 1 & 3 \\ 4 & 1 & 6 \\ 7 & 1 & 9 \end{vmatrix}$$

$$C3 \rightarrow C3 - C1 \Rightarrow \begin{vmatrix} 1 & 1 & 2 \\ 4 & 1 & 2 \\ 7 & 1 & 2 \end{vmatrix} = 0 \text{ , as two columns are equal}$$

(iv) Use of the Factor theorem

To factorise $\Delta = \begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$:

Δ can be considered as a polynomial in x (for example); $f(x)$ say.

As $f(y) = \begin{vmatrix} 1 & y & y^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = 0$ (as rows 1 & 2 are equal), it follows

that $(x - y)$ is a factor of Δ.

Similarly $(x - z)$ is a factor, and also $(y - z)$ (by symmetry: considering Δ as a polynomial in y or z).

We can write $\Delta = a(x-y)(x-z)(y-z)$. There won't be any other factors involving $x, y \text{ or } z$, since if we expand $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$ by the 1st column, all the terms are seen to be cubic in x, y & z.

Then, as one of the terms of the expansion is $x^2(-y)$ [expanding by the 1st row, for example], a must be -1,

and $\Delta = -(x-y)(x-z)(y-z)$, or $\Delta = (x-y)(y-z)(z-x)$

(v) Alternative method (when the elements are numbers)

The matrix is extended, by adding on repeats of the 2nd and 3rd rows, as shown below. For each upward or downward diagonal

with 3 elements, the elements are multiplied together, and the sign of the resulting product is reversed in the case of the downward diagonals.

Example: To find $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$

1   2   3
4   5   6
7   8   9

1   2   3
4   5   6

$\Delta = (7)(5)(3) + (1)(8)(6) + (4)(2)(9)$

$-(1)(5)(9) - (4)(8)(3) - (7)(2)(6)$

$= 105 + 48 + 72 - 45 - 96 - 84 = 0$

(3) Cramer's rule

This provides a compact way of solving simultaneous equations. For two equations, expressed in matrix form as

$\begin{pmatrix} a & c \\ b & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$, the solution is:

$$x = \frac{\begin{vmatrix} e & c \\ f & d \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}} \ \& \ y = \frac{\begin{vmatrix} a & e \\ b & f \end{vmatrix}}{\begin{vmatrix} a & c \\ b & d \end{vmatrix}}$$

and the same method applies to larger numbers of equations.

General Proof

Let the system of equations be:

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

...

$a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n$

Let A be the matrix $[a_{ij}]$ and let c_{ij} be the cofactor of a_{ij} in A.

Then, multiplying the 1st equation by c_{11}, the 2nd equation by c_{21}, ..., the last equation by c_{n1} , and adding:

$$x_1 \sum_{i=1}^{n} a_{i1}c_{i1} + x_2 \sum_{i=1}^{n} a_{i2}c_{i1} + \cdots + x_n \sum_{i=1}^{n} a_{in}c_{i1} = \sum_{i=1}^{n} b_i c_{i1}$$

But $\sum_{i=1}^{n} a_{i1}c_{i1} = |A|$ (expanding by the 1st column),

and $\sum_{i=1}^{n} a_{i2}c_{i1} = 0$ (property of alien cofactors: applying the cofactors of the 1st column to the elements of the 2nd column) etc., whilst $\sum_{i=1}^{n} b_i c_{i1}$ is the determinant of the matrix formed by replacing the 1st column of A by the column of $b_i s$ (and expanding by the 1st column).

Thus $x_1|A| = \begin{vmatrix} b_1 & a_{12} & \dots a_{1n} \\ b_2 & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ b_n & a_{n2} & \dots a_{nn} \end{vmatrix}$

and hence $x_1 = \dfrac{1}{|A|} \begin{vmatrix} b_1 & a_{12} & \dots a_{1n} \\ b_2 & a_{22} & \dots a_{2n} \\ \dots & \dots & \dots \\ b_n & a_{n2} & \dots a_{nn} \end{vmatrix}$

(and similarly for $x_2, \dots, x_n$)

(B) Transposes

[denoted by A^T or A']

(i) $(AB)^T = B^T A^T$

Note that A & B don't need to be square matrices (provided that AB is defined).

(ii) AA^T is symmetric

as $(AA^T)^T = (A^T)^T A^T = AA^T$

(iii) $(A^T)^{-1} = (A^{-1})^T$ (see Matrices - Exs (General), Q5)

(iv) $|A^T| = |A|$

(v) A is described as orthogonal if $A^{-1} = A^T$

(vi) Scalar products

If $\underline{x}_1$ & $\underline{x}_2$ are two column vectors, their scalar product is given by

$\underline{x}_1{}^T \underline{x}_2$ (or $\underline{x}_2{}^T \underline{x}_1$; the effect of taking the transpose of $\underline{x}_1{}^T \underline{x}_2$, which (as it is a scalar) leaves it unchanged); ie a row vector multiplied (on its right) by a column vector. [This gives a

$(1 \times n) \times (n \times 1) = 1 \times 1$ matrix; ie a scalar. The reverse order would be incorrect, giving a

$(n \times 1) \times (1 \times n) = n \times n$ matrix.]

(C) 3×3 Inverses

Let $|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{vmatrix}$, for example

$= 1(1 \times 3 - 2 \times 4) - 5(3 \times 3 - 2 \times 2) + 6(3 \times 4 - 1 \times 2)$ (1)

(expanding by the 1st column).

Let $A_1 = (1 \times 3 - 2 \times 4)$, $A_2 = -(3 \times 3 - 2 \times 2)$,

$A_3 = (3 \times 4 - 1 \times 2)$

or $A_1 = + \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, $A_2 = - \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix}$, $A_3 = + \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$ in general.

These are referred to as the **cofactors** of a_1, a_2, a_3, respectively.

Other cofactors are defined similarly, with the $+$ and $-$ signs alternating as we go round the matrix. Thus:

$B_1 = - \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$, $B_2 = + \begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$, $B_3 = - \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

$C_1 = + \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$, $C_2 = - \begin{vmatrix} a_1 & b_1 \\ a_3 & b_3 \end{vmatrix}$, $C_3 = + \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

The cofactors are thus **signed** 2×2 determinants. The **unsigned** 2×2 determinants are referred to as the **minors** of a_1, a_2, a_3 respectively.

So, referring back to (1), $a_1 A_1 + a_2 A_2 + a_3 A_3 = |A|$

Similarly it can be shown that

$b_1 B_1 + b_2 B_2 + b_3 B_3 = |A|$ and $c_1 C_1 + c_2 C_2 + c_3 C_3 = |A|$

Also $b_1 A_1 + b_2 A_2 + b_3 A_3 =$

$3(1 \times 3 - 2 \times 4) - 1(3 \times 3 - 2 \times 2) + 2(3 \times 4 - 1 \times 2)$

$= \begin{vmatrix} 3 & 3 & 2 \\ 1 & 1 & 4 \\ 2 & 2 & 3 \end{vmatrix}$ (expanding by the 1st column) $= 0$

(as two of the columns of the determinant are identical).

[Note that the 1st column of $a_i s$ in the original matrix has been replaced with the column of $b_i s$.]

Thus $b_1 A_1 + b_2 A_2 + b_3 A_3 = 0$ (property of 'alien cofactors'),

and similarly for other combinations of letters, so that

eg $c_1 B_1 + c_2 B_2 + c_3 B_3 = 0$

Now consider the matrix of cofactors: $\begin{pmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{pmatrix}$

and form the transpose: $\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

This transpose is called the **adjugate** or **adjoint** of the original matrix.

Then, from the above, it follows that

$$\begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$$

$$= |A| \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence $A^{-1} = \frac{1}{|A|} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix}$

Summary of Procedure

(1) Find cofactors of each cell (**signed** 2×2 determinants).

(2) Create the **transpose** of the matrix of cofactors.

(3) Divide by |A|.

Exercise 1: Find $\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}^{-1}$

Solution

Step 1: Matrix of cofactors $= \begin{pmatrix} -5 & 9 & 4 \\ -5 & -9 & 16 \\ 10 & 6 & -14 \end{pmatrix}$

Step 2: Transpose of matrix of cofactors $= \begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix}$

Step 3: $\begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}^{-1} = \frac{1}{30} \begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix}$

[As a check, $\frac{1}{30} \begin{pmatrix} -5 & -5 & 10 \\ 9 & -9 & 6 \\ 4 & 16 & -14 \end{pmatrix} \begin{pmatrix} 1 & 3 & 2 \\ 5 & 1 & 4 \\ 6 & 2 & 3 \end{pmatrix}$ should equal $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$]

Exercise 2

Show that $\begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \begin{pmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{pmatrix} = \begin{pmatrix} |A| & 0 & 0 \\ 0 & |A| & 0 \\ 0 & 0 & |A| \end{pmatrix}$

Solution

$$a_1 A_1 + b_1 B_1 + c_1 C_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (expanding by 1st row) } = |A|$$

$$a_1 A_2 + b_1 B_2 + c_1 C_2 = - \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ (expanding by 1st row) } =$$

0,

as two of the rows are identical; and similarly for other cells.