Differential Equations - Oscillations (11 pages; 29/5/20)

(A) Simple Harmonic Motion (SHM)

(1) Example 1: Hanging spring

By Newton's 2nd law, $m\frac{d^2x}{dt^2} = mg - T$

where the tension $T = k(e + x)$, with k being the stiffness of the spring, e the extension of the spring hanging in equilibrium, and x the further extension beyond the equilbrium position.

As equilibrium gives $mg = ke$, the differential equation reduces to $\frac{d^2x}{dt^2} = -\frac{kx}{m}$

(2) Example 2: Pendulum

The distance travelled along the path of the pendulum is $l\theta$, where l is the length of the pendulum and θ is the angle (in radians) that it makes with the vertical.

The acceleration along the path is then $l\frac{d^2\theta}{dt^2}$ and the component of the bob's weight along the path is $-mgsin\theta$.

So $ml\frac{d^2\theta}{dt^2} = -mgsin\theta$ and, with $sin\theta \approx \theta$,

this becomes $\frac{d^2\theta}{dt^2} \approx -\frac{g\theta}{l}$

(3) The above examples are both cases of simple harmonic motion, with the general equation $\frac{d^2x}{dt^2} = -\omega^2 x$

[The term ω^2 is chosen, to represent a positive quantity.]

Referring to "Second-order Differential Equations", the general solution is obtained by considering the auxiliary equation

$\lambda^2 + \omega^2 = 0$, giving $\lambda = \pm\omega i$,

so that $x = Ae^{i\omega t} + Be^{-i\omega t}$

$= (Acos\omega t + iAsin\omega t) + (Bcos\omega t - iBsin\omega t)$

$= Ccos\omega t + Dsin\omega t$,

where $C = A + B$ and $D = (A - B)i$

As C and D must be real (to give a solution in the real world), we require that A and B be complex conjugates of each other.

Then $Ccos\omega t + Dsin\omega t$ can be written as $asin(\omega t + \alpha)$,

where a is the **amplitude** of the oscillations, and the **period** T is given by $\omega T = 2\pi$, so that $T = \frac{2\pi}{\omega}$, and the **frequency** (the number of cycles in 1 second) $= \frac{1}{T} = \frac{\omega}{2\pi}$; ω is the '**angular frequency'.**

Note: $\dot{x}$ and $\ddot{x}$ may be used to represent $\frac{dx}{dt}$ and $\frac{d^2x}{dt^2}$, respectively.

(4) The equation $\frac{d^2x}{dt^2} = -\omega^2 x$ can also be solved as follows:

$\frac{d^2x}{dt^2} = \frac{dv}{dt} = \frac{dv}{dx} \cdot \frac{dx}{dt} = v\frac{dv}{dx}$; so $v\frac{dv}{dx} = -\omega^2 x$

and hence $\int v\, dv = -\, \omega^2 \int x\, dx$,

giving $\frac{1}{2}v^2 = -\frac{1}{2}\omega^2 x^2 + C$

Then, if $x = a$ when $v = 0$,

$0 = -\frac{1}{2}\omega^2 a^2 + C$, so that $C = \frac{1}{2}\omega^2 a^2$

and $v^2 = \omega^2(a^2 - x^2)$

Then x can be determined by writing $v = \dot{x}$ and integrating again:

$\frac{dx}{dt} = \pm\omega\sqrt{a^2 - x^2} \Rightarrow \pm\int \frac{1}{\sqrt{a^2-x^2}} = \omega\int dt$

$\Rightarrow \omega t = arcsin\left(\frac{x}{a}\right) - \alpha$ or $arccos\left(\frac{x}{a}\right) - \alpha$

$\Rightarrow x = asin(\omega t + \alpha)$ or $x = acos(\omega t + \alpha)$

Note though that $acos(\omega t + \alpha) = acos(-[\omega t + \alpha])$

$= asin\left(\frac{\pi}{2} - (-[\omega t + \alpha])\right) = asin(\omega t + \alpha')$,

where $\alpha' = \alpha + \frac{\pi}{2}$ (or just from the fact that the sine function lags behind the cosine function, so that eg $\cos 0 = \sin\left(\frac{\pi}{2}\right)$).

(5) Example: Consider an elastic string AB of natural length 2.5 and modulus of elasticity 15.

A particle P of mass 0.5 is attached to the string such that, when it is unstretched, $AP = 1$ and $PB = 1.5$

The string with P attached is then stretched horizontally, so that A and B are a distance 5 apart.

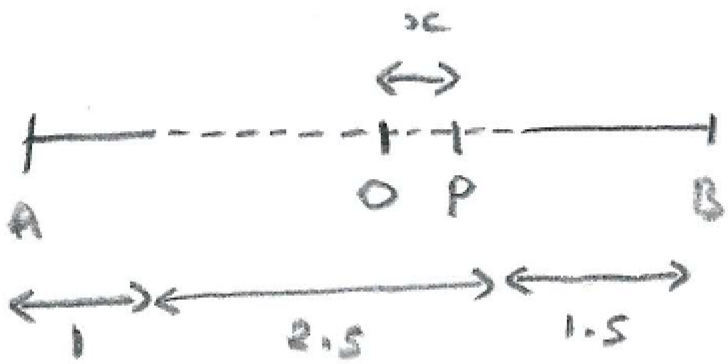
P is then pulled to one side, so that $PB = 2$, and then released.

Describe the subsequent motion of P.

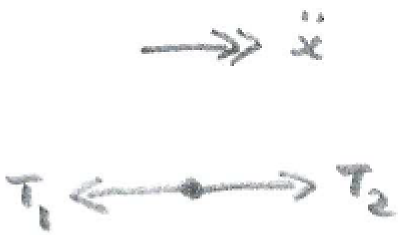
Solution

We can consider the string AB as consisting of two strings AP and PB, each with modulus of elasticity 15, and natural lengths 1 and 1.5, respectively.

First of all we need to consider the general position of P during the motion. Suppose that it is at a distance x to the right of O, which is halfway between A and B.



We can create a force diagram for P, as shown below, where T_1 & T_2 are the tensions in AP and PB, respectively.



Then, by N2L, $T_2 - T_1 = 0.5\ddot{x}$,

and by Hooke's law, $T_1 = \frac{15(2.5+x-1)}{1} = 15(1.5 + x)$

and $T_2 = \frac{15(2.5-x-1.5)}{1.5} = 10(1-x)$

and so $10(1-x) - 15(1.5+x) = 0.5\ddot{x}$

$\Rightarrow -12.5 - 25x = 0.5\ddot{x}$

$\Rightarrow \ddot{x} + 50x = -25 \;\; (1)$

Note that this isn't in the standard SHM form $\ddot{x} + \omega^2 x = 0$

However, if we write $y = x + 0.5$, then $\ddot{y} + 50y = 0$.

This shows that there is SHM about the point where $y = 0$; ie where $x = -0.5$

Alternatively (for the practice), (1) can be solved as $x = CF + PI$

['complementary function' and 'particular integral' - see "Differential equations: 2nd order (linear, constant coeffs)"]

The auxiliary equation for the homogeneous equation

$\ddot{x} + 50x = 0$ is $\lambda^2 + 50 = 0$, giving $\lambda = \pm i\sqrt{50}$,

so that $CF = Ae^{i\sqrt{50}\,t} + Be^{-i\sqrt{50}\,t}$

$= \left(Acos\sqrt{50}t + iAsin\sqrt{50}t\right) + \left(Bcos\sqrt{50}t - iBsin\sqrt{50}t\right)$

$= Ccos\sqrt{50}t + Dsin\sqrt{50}t$,

where $C = A + B$ and $D = (A - B)i$

As C and D must be real (to give a solution in the real world), we require that A and B be complex conjugates of each other.

Then $Ccos\sqrt{50}t + Dsin\sqrt{50}t$ can be written as $Esin(\sqrt{50}t + \alpha)$

For the PI, the trial function is $x = a$, and (1) then gives

$50a = -25$, so that $a = -0.5$

So the general solution is $x = Esin\left(\sqrt{50}t + \alpha\right) - 0.5$

(ie SHM about $x = -0.5$, as found previously).

Applying the initial conditions,

$t = 0, x = 0.5 \Rightarrow 0.5 = Esin\alpha - 0.5 \Rightarrow Esin\alpha = 1$

Also $\dot{x} = E\sqrt{50}\cos(\sqrt{50}t + \alpha)$,

so that $t = 0, \dot{x} = 0 \Rightarrow 0 = E\sqrt{50}\cos\alpha \Rightarrow \alpha = \frac{\pi}{2}$

Then $Esin\alpha = 1 \Rightarrow E = 1$,

so that $x = sin\left(\sqrt{50}t + \frac{\pi}{2}\right) - 0.5$ (2)

ie SHM about $x = -0.5$, with amplitude 1 and period $\frac{2\pi}{\sqrt{50}}$

[Either consider T such that $\sqrt{50}\,T = 2\pi$, or note that $sin(\sqrt{50}t)$ is obtained from $sin\ t$ by a stretch of scale factor $\frac{1}{\sqrt{50}}$, resulting in the period being reduced by this factor.]

As a check, we can also confirm that when $x = -0.5$,

$T_1 = \frac{15(2-1)}{1} = 15$ and $T_2 = \frac{15(3-1.5)}{1.5} = 15$, so that the net force, and hence acceleration, is zero when P is at the centre of the SHM.

Also, note that (2) $\Rightarrow$ $-1.5 \leq x \leq 0.5$, and that the strings are taut within this range (so that Hooke's law applies throughout).

[Had the initial condition been that $x = 1$ when $t = 0$, then we would have the solution $x = 1.5sin\left(\sqrt{50}t + \frac{\pi}{2}\right) - 0.5$, but this would imply that $-2 \leq x \leq 1$, and the string AP would not be taut for $-2 \leq x < -1.5$, so that the equation of motion is invalidated.

(B) Other situations

(1) The examples in (A) involved differential equations of the form $\ddot{x} + \omega^2 x = c$, where c is a constant, and the motion was seen to be SHM (whether $c = 0$ or not).

We shall now look at equations of the following forms:

(I) Friction: $\ddot{x} + \omega^2 x = \pm\mu g$, where the sign depends on the direction of motion

(II) Damping: $\ddot{x} + \alpha\dot{x} + \omega^2 x = 0$, where $\alpha > 0$

(III) $\ddot{x} + \alpha\dot{x} + \omega^2 x = c \quad (\alpha > 0)$

(IV) Forced oscillations: $\ddot{x} + \alpha\dot{x} + \omega^2 x = csin\Omega t \quad (\alpha > 0)$

(2) In the case of (I), where there is a frictional force,

$m\ddot{x} = -m\omega^2 x - \mu mg$ when motion is in the direction of x increasing,

and $m\ddot{x} = -m\omega^2 x + \mu mg$ when motion is in the direction of x decreasing.

So the graph takes a complicated form.

(3) Damping

In the case of (II), where $\ddot{x} + \alpha\dot{x} + \omega^2 x = 0$ (with $\alpha > 0$)

or $\ddot{x} = -\alpha\dot{x} - \omega^2 x$, the presence of the $\alpha\dot{x}$ term indicates that the acceleration is dependent on the velocity, and this happens where there is 'damping', such as is caused by movement through a viscous liquid.

(3.1) Heavy damping (or 'overdamping')

If the discriminant of the auxiliary equation, $\alpha^2 - 4\omega^2 > 0$, then the auxiliary equation has real roots, so that the solution is of the form $x = Ae^{-pt} + Be^{-qt}$ (p & $q > 0$), and the system decays without oscillating.

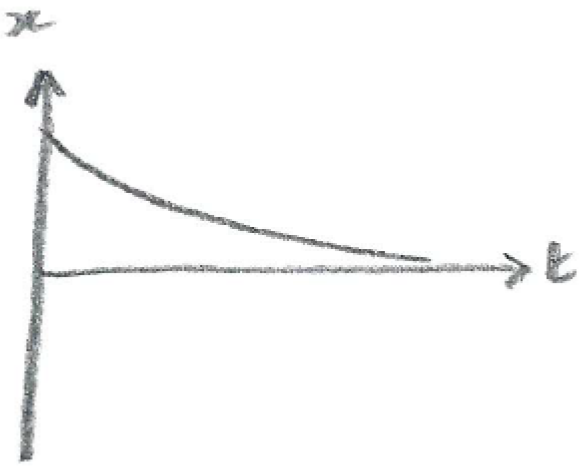
[The roots of the auxiliary equation are $\frac{-\alpha \pm \sqrt{\alpha^2 - 4\omega^2}}{2}$ and are thus both negative, as $\alpha > 0$.]

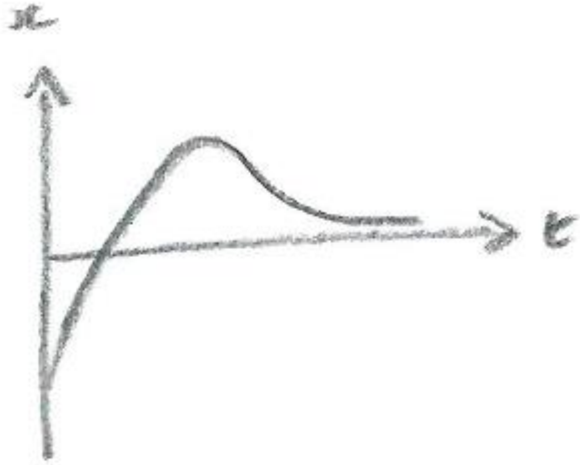
If exactly one of A and B is negative, then the graph will cross the t-axis once:

$Ae^{-pt} + Be^{-qt} = 0 \Rightarrow e^{-pt}\left(A + Be^{(p-q)t}\right) = 0$

$\Rightarrow e^{(p-q)t} = -\frac{A}{B}$

Typical graphs are shown below.





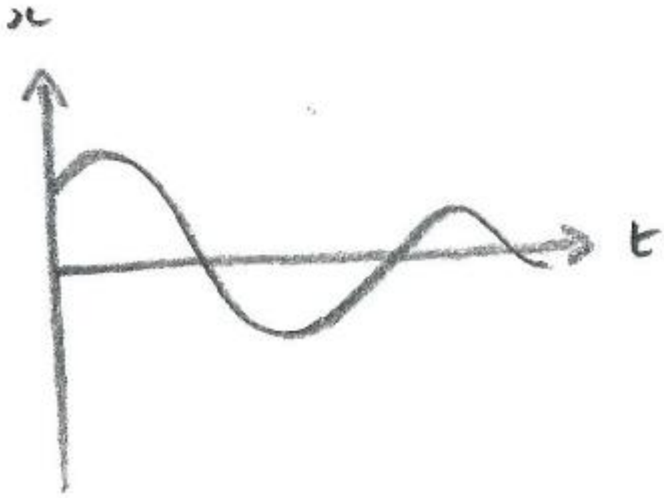
(3.2) Critical damping

If the discriminant of the auxiliary equation, $\alpha^2 - 4\omega^2 = 0$, then the auxiliary equation has a repeated root, so that the solution is of the form $(A + Bt)e^{-\frac{\alpha}{2}t}$, and the system decays without oscillating, in a similar way to heavy damping.

Once again, if exactly one of A and B is negative, then the graph will cross the t-axis once, when $t = -\frac{A}{B}$.

(3.3) Light damping (or 'underdamping')

If the discriminant of the auxiliary equation, $\alpha^2 - 4\omega^2 < 0$, then the auxiliary equation has complex roots, so that the solution is of the form $Ae^{-\frac{\alpha}{2}t}\sin(bt + \varepsilon)$, and the system oscillates whilst decaying.



(3.4) In summary, if the roots of the auxiliary equation are real, then the system decays without oscillating (heavy and critical damping); and if they are complex then the system oscillates whilst decaying (light damping).

Note that the damping force is $m\alpha\dot{x}$, whilst the restoring force is $m\omega^2 x$, and so the relative sizes of α and ω^2 determine whether the damping is heavy (when $\alpha^2 > 4\omega^2$) or light (when $\alpha^2 < 4\omega^2$).

The initial conditions will determine the values of the constants in the solution, and hence the precise pattern of the graphs.

(4.1) In the case of (III), where $\ddot{x} + \alpha\dot{x} + \omega^2 x = c$, the substitution $y = \omega^2 x - c$ can be made, to give

$\frac{1}{\omega^2}\ddot{y} + \alpha\frac{1}{\omega^2}\dot{y} + y = 0$ or $\ddot{y} + \alpha\dot{y} + \omega^2 y = 0$

This is similar to the example of the SHM in (A)(5), where the constant term c resulted from the fact that x wasn't being measured from its centre of oscillation.

(4.2) Alternatively, if $\ddot{x} + \alpha\dot{x} + \omega^2 x = c$ is written as

$\ddot{x} + \alpha\dot{x} + \omega^2 x = \alpha u$, this can represent a situation where the particle is moving at constant speed u at $x = 0$ (ie with (instantaneously) zero acceleration)[as this satisfies the differential equation]

(5.1) Forced oscillations

(IV) can be rewritten as $\ddot{x} = -\omega^2 x - \alpha\dot{x} - csin\Omega t$

The presence of the $csin\Omega t$ term indicates that there is some external force that oscillates with time.

The particular integral will be of the form $asin(\Omega t + \varepsilon)$, and this will be superimposed on the appropriate damped solution.

(5.2) When there is no damping term, so that

$\ddot{x} = -\omega^2 x - csin\Omega t$ (or if the damping term is small) then the phenomenon of resonance can be observed when $\Omega = \omega$. This involves the build up of increasingly large oscillations.

In this case the usual trial function for the particular integral is

$x = psin\omega t + qcos\omega t$, which is the complementary function. (This doesn't happen when there is damping, as this gives rise to a negative exponential factor in the complementary function.)

In such cases, the trial function is $x = t(psin\omega t + qcos\omega t)$, and so the resulting particular integral will increase in amplitude with time.

There are known cases of structures such as bridges collapsing due to forced oscillations (eg caused by wind) having the same frequency as the 'natural frequency' ω of the structure.

(6) Situation involving two particles

See STEP 2014, P3, Q10 for an example of a conservative system involving SHM, which (surprisingly) doesn't return to its starting position.