Differential Equations: Approximate methods

(9 pages; 26/2/21)

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(1) Tangent Fields

Whether or not an **analytical** (ie non-approximate) solution exists for a differential equation of the form $\frac{dy}{dx} = f(x, y)$, it will be possible to plot the **direction indicators** for the curve.

Example 1: $\frac{dy}{dx} = x + y$

Figure 1 below shows the direction indicators at various points, whilst Figure 2 shows the family of solutions of the equation. (This can be shown to be $y = Ae^x - x - 1$.)

An **isocline** is a locus of points for which the direction indicators are the same. Here, for example, the line $y = -x$ is an isoscline where the gradient of the direction indicator is 0.

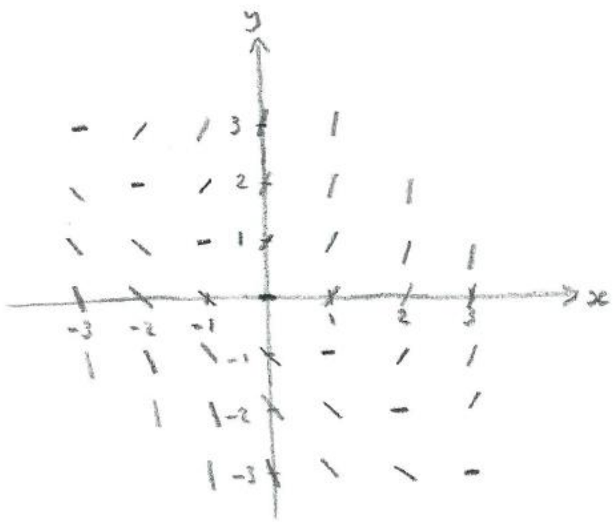


Figure 1

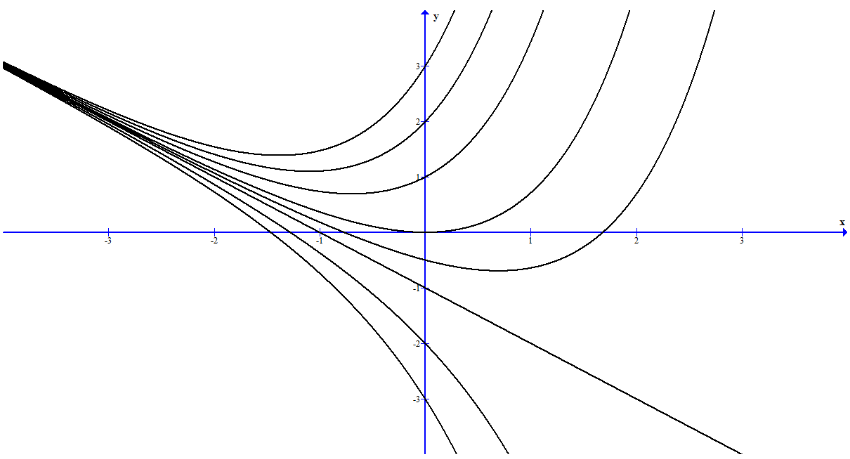
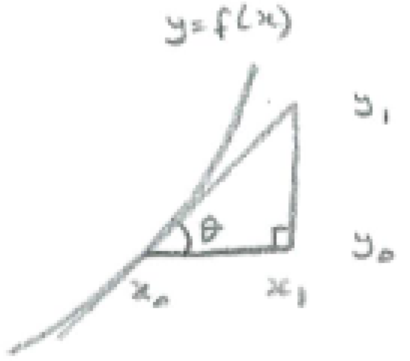


Figure 2

(2) **Euler's Method** (for 1st order equations)

Consider the differential eq'n $\frac{dy}{dx} = g(x, y)$, with solution $y = f(x)$.

Consider the diagram below, where the tangent to $y = f(x)$ is drawn at the point (x_0, y_0).



We will use y_1 as an estimate for $f(x_0 + h) = f(x_1)$

(The greater the change of gradient at the point - ie the greater the magnitude of $f''(x_0)$ - the less accurate the estimate will be.)

Thus $f(x_1) \approx y_1 = y_0 + htan\theta = y_0 + hf'(x_0) = y_0 + hg(x_0, y_0)$

[$tan\theta$ can be thought of as the scale factor to be applied to the base of the triangle, in order to obtain its height]

So $y_1 = y_0 + hg(x_0, y_0)$

Then we can write $f(x_2) = f(x_1 + h) \approx y_2 = y_1 + hg(x_1, y_1)$,

and so on for $f(x_3)$ etc.

But note that, this time, the point (x_1, y_1) will not actually be on the curve – which introduces a further approximation into the sequence of estimates y_1 , y_2 , y_3 , ... (Note also that y_2 is being defined as $y_1 + hg(x_1, y_1)$; ie there is no approximation involved.)

Example 2: $\frac{dy}{dx} = x + y$, where $y = 0$ when $x = 0$

With $x_0 = 0$, $y_0 = 0$ and $h = 0.1$,

$x_1 = 0.1$, $y_1 = 0 + 0.1(0 + 0) = 0$

$x_2 = 0.2,\ y_2 = 0 + 0.1(0.1 + 0) = 0.01$

$x_3 = 0.3,\ y_3 = 0.01 + 0.1(0.2 + 0.01) = 0.031$

(3) Improved estimate for Euler's method

If α is a particular value of x , then, for small values of h, it can be shown that the estimate of y at $x = \alpha$, $y(\alpha)$ is approximately a linear function of h; i.e. $y(\alpha) \approx mh + c$ (*)

By carrying out Euler's method for two values of h, and obtaining a value for $y(\alpha)$ in each case, two simultaneous equations of the form (*) are created, and these can be solved to obtain a value for c. This value is then equivalent to putting $h = 0$, and is thus an improved value for $y(\alpha)$.

For Example 2, with $h = 0.05$,

$x_1 = 0.05,\ \ y_1 = 0 + 0.05(0 + 0) = 0$

$x_2 = 0.1,\ y_2 = 0 + 0.05(0.05 + 0) = 0.0025$

$x_3 = 0.15,\ y_3 = 0.0025 + 0.05(0.1 + 0.0025) = 0.007625$

$x_4 = 0.2,\ y_4 = 0.007625 + 0.05(0.15 + 0.007625) = 0.01550625$

Thus, with $\alpha = 0.2$ and $h = 0.05$, an estimate for $y(0.2)$ is 0.01550625 or 0.0155 (3sf)

We can then write $0.01550625 = m(0.05) + c$ (1)

Earlier we obtained the estimate of 0.01 for $y(0.2)$, with $h = 0.1$,

and this gives $0.01 = m(0.1) + c$ (2)

Then $2 \times (1) - (2)$ gives $c = 0.0310125 - 0.01 = 0.0210125$, and hence an improved estimate for $y(0.2)$ is 0.0210 (3sf)

The true value is $e^{0.2} - 0.2 - 1 = 0.0214$ (3sf).

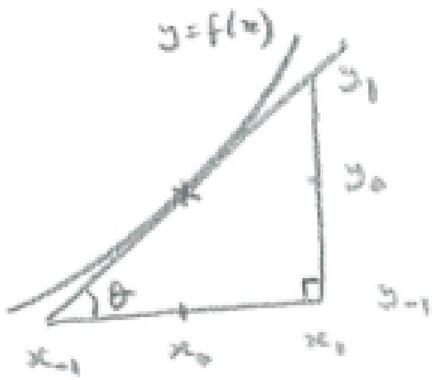
To summarise:

	estimate of $y(0.2)$
$h = 0.1$	0.01
$h = 0.05$	0.0155
$h \approx 0$	0.0210
	0.0214 (true value)

(4) Midpoint method (for 1st order equations)

As before, consider the differential eq'n $\frac{dy}{dx} = g(x, y)$, with solution $y = f(x)$.

An improvement can usually be made to Euler's method by considering approximations to the y-coordinate either side of y_0 (see the diagram below).



We can write $f(x_1) \approx y_1 = y_{-1} + 2h tan\theta = y_{-1} + 2hf'(x_0)$

$= y_{-1} + 2hg(x_0, y_0)$

and $f(x_2) \approx y_2 = y_0 + 2hg(x_1, y_1)$ etc

Euler's method is commonly used to obtain y_1, and get the mid-point method started with y_2 (so that $y_1 = y_{-1} + 2hg(x_0, y_0)$ is not used).

Example 2 (again): $\frac{dy}{dx} = x + y$, where $y = 0$ when $x = 0$

with $h = 0.1$ again.

Solution

$x_0 = 0$, $y_0 = 0$ again.

$x_1 = 0 + 0.1 = 0.1$

Euler's method is applied to find y_1, to give $y_1 = 0$, as before.

Then $\frac{dy}{dx}|_1 \approx x_1 + y_1 = 0.1 + 0 = 0.1$

$x_2 = 0.1 + 0.1 = 0.2$

By the midpoint formula, $y_2 = y_0 + 2h(x_1 + y_1)$,

so that $y_2 \approx 0 + 2(0.1)(0.1) = 0.02$

$x_3 = 0.2 + 0.1 = 0.3$

Then $\frac{dy}{dx}|_2 \approx x_2 + y_2 = 0.2 + 0.02 = 0.22$

$y_3 \approx y_1 + 2h(x_2 + y_2)$,

so that $y_3 \approx 0 + 2(0.1)(0.22) = 0.044$

[This compares with 0.031 by Euler's method.]

(5) 2nd Order method

Consider the differential eq'n $\frac{d^2y}{dx^2} = g(x, y)$, with solution

$y = f(x)$.

The following iterative formula can be derived

$$\boldsymbol{y_r = -y_{r-2} + 2y_{r-1} + h^2 \mathrm{g}(x_{r-1}, y_{r-1})}$$

Derivation: As $\frac{d^2y}{dx^2}$ is the gradient of $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}|_0 \approx \frac{\frac{dy}{dx}|_0 - \frac{dy}{dx}|_{-1}}{h}$

[Notice that we are looking backwards this time, whereas Euler's method looks forward with $\frac{dy}{dx}|_0 = \frac{y_1 - y_0}{h}$]

$$= \frac{\left(\frac{y_1 - y_0}{h}\right) - \left(\frac{y_0 - y_{-1}}{h}\right)}{h} = \frac{(y_1 - y_0) - (y_0 - y_{-1})}{h^2} = \frac{y_1 - 2y_0 + y_{-1}}{h^2}$$

$$\Rightarrow h^2 \frac{d^2y}{dx^2}|_0 \approx y_1 - 2y_0 + y_{-1}$$

$$\Rightarrow y_1 \approx -y_{-1} + 2y_0 + h^2 \frac{d^2y}{dx^2}|_0 \quad (*)$$

which gives the required iterative formula, if we re-define y_r by

$y_r = -y_{r-2} + 2y_{r-1} + h^2 \mathrm{g}(x_{r-1}, y_{r-1})$

[The y_i are the values from Euler's method, and are related by the approximate relation (*), but we are at liberty to define them as we wish for the 2nd Order method.]

Thus $y_2 = -y_0 + 2y_1 + h^2 \mathrm{g}(x_1, y_1)$

Once again, Euler's method will often be applied to find y_1. But, in order to do this, a value will need to have been provided for $f'(x_0)$ [in order to use $y_1 = y_0 + hf'(x_0)$]

Example 3: $\frac{d^2y}{dx^2} = x(x+y)$, given that when $x = 1, y = 2$ and $\frac{dy}{dx} = 1$; with $h = 0.1$

Solution

Method 1 (using Euler's formula)

$x_0 = 1\ ,\ y_0 = 2$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

Use Euler's method to obtain a value for y_1:

$y_1 = y_0 + h\frac{dy}{dx}|_0 = 2 + (0.1)(1) = 2.2$

$x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

Then the formula $y_r = -y_{r-2} + 2y_{r-1} + h^2 g(x_{r-1} + y_{r-1})$, gives:

$y_2 = -y_0 + 2y_1 + h^2 x_1(x_1 + y_1)$

$= -2 + 2(2.2) + (0.01)(1.1)(1.1 + 2.2)$

$= 2.4363$

and values for y_3 etc are obtained in the same way.

Method 2 (a more accurate - but longer - approach, using the midpoint formula)

$x_0 = 1\ ,\ y_0 = 2$

$x_1 = x_0 + h = 1 + 0.1 = 1.1$

The 2nd order formula $y_r = -y_{r-2} + 2y_{r-1} + h^2 g(x_{r-1} + y_{r-1})$ gives

$y_1 = -y_{-1} + 2y_0 + h^2 x_0(x_0 + y_0)$

so that $y_1 = -y_{-1} + 2(2) + (0.01)(1)(1+2)$

$= -y_{-1} + 4.03$ (1),

whilst the midpoint formula gives $y_1 = y_{-1} + 2h\frac{dy}{dx}|_0$

so that $y_1 = y_{-1} + 2(0.1)(1) = y_{-1} + 0.2$ (2)

Adding (1) & (2): $2y_1 = 4.23$ and $y_1 = 2.115$

[compared with 2.2 by Method 1]

Then $y_2 = -y_0 + 2y_1 + h^2 g(x_1 + y_1)$

$= -y_0 + 2y_1 + h^2 x_1(x_1 + y_1)$

$= -2 + 2(2.115) + (0.01)(1.1)(1.1 + 2.115)$

$= 2.265365$ [compared with 2.4363 by Method 1]

(6) General Points

(i) Values of y_n shouldn't be given to too many decimal places (though a reasonably large number of dps should be kept in the intermediate calculations).

If accurate values of $f(x_n)$ are required, then the process can be repeated for smaller h, until no further change occurs, to the required number of decimal places.

(ii) Euler's method can be used (eg to find another value of y_n) whenever either (a) a formula is provided for $\frac{dy}{dx}$, or (b) when a value is given for $\frac{dy}{dx}$ for a particular value of x.