Cubic Graphs - Exercises (Solutions) (10 pages; 29/7/16)

(1) Point of Inflexion (or 'inflection')

This can be defined as a turning point of the gradient.

So $\frac{d}{dx}\left(\frac{dy}{dx}\right) = 0$ and $\frac{d^2}{dx^2}\left(\frac{dy}{dx}\right) \neq 0$

(sufficient but not necessary condition)

ie $\frac{d^2y}{dx^2} = 0$ & $\frac{d^3y}{dx^3} \neq 0$

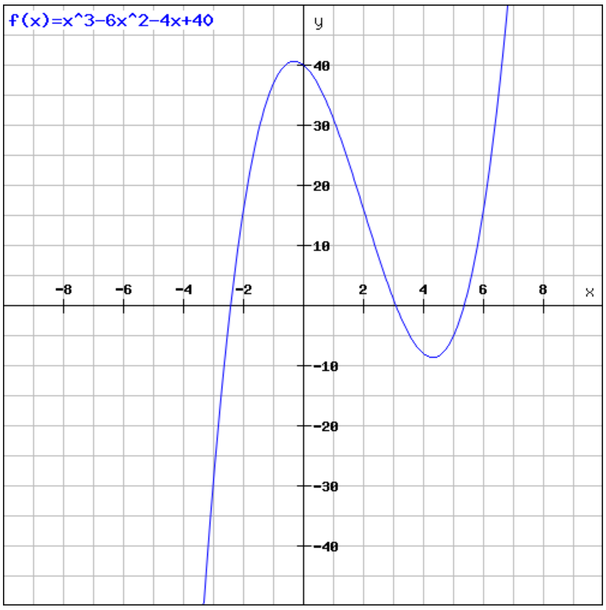


Fig. 1: $p(x) = x^3 - 6x^2 - 4x + 40$

Point of Inflexion at (2, 16)

(2) For $f(x) = ax^3 + bx^2 + cx + d$, what is the x-coordinate of the PoI?

Solution

$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$

$f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

(3) Give examples of cubic functions for which the PoI is at the Origin, and the gradient at the Origin is (a) 1 (b) -1. How do the shapes of the two graphs differ?

Solution

As the graph passes through the Origin, we can consider

$f(x) = ax^3 + bx^2 + cx$

From (2), PoI is at $x = -\frac{b}{3a}$, so that $b = 0$

$f'(x) = 3ax^2 + c$

$f'(0) = 1 \Rightarrow c = 1$

eg $y = 2x^3 + x = x(2x^2 + 1)$

$f'(0) = -1 \Rightarrow c = -1$

eg $y = 2x^3 - x = x(2x^2 - 1)$

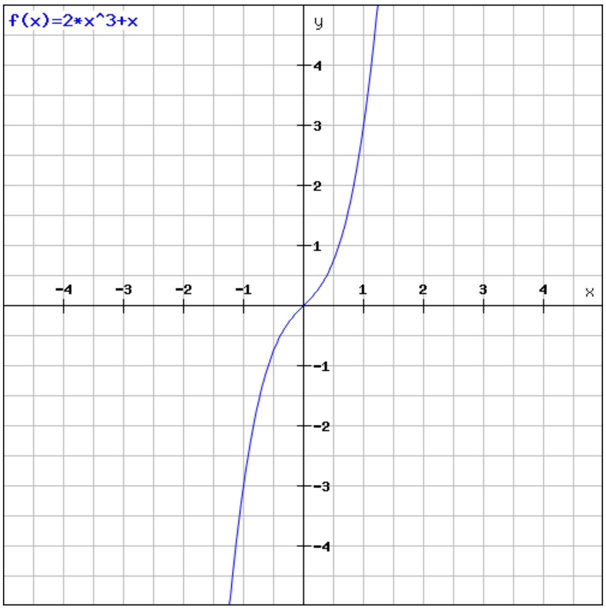


Fig. 2: $y = 2x^3 + x$

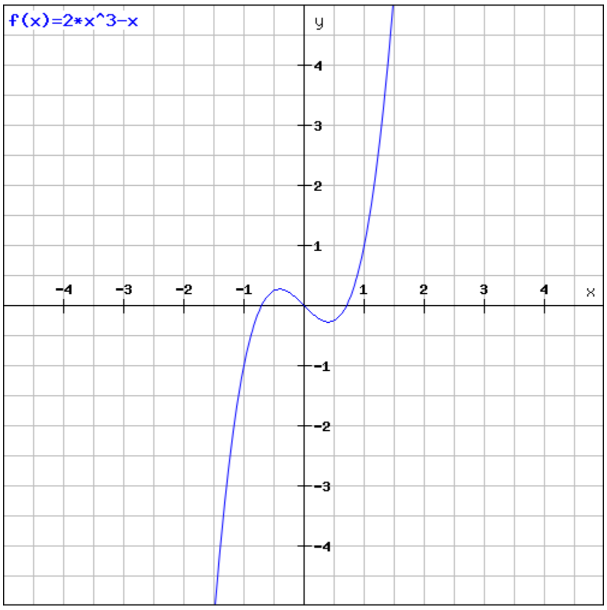


Fig. 3: $y = 2x^3 - x$

(4) Compare the x-coordinate of the PoI with those of the turning points (where they exist).

Solution

If $f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f'(x) = 0 \Rightarrow x = \frac{-2b \pm \sqrt{4b^2 - 12ac}}{6a}$

From (2), the PoI is at $x = -\frac{b}{3a}$; ie the mid-point of the turning points (if they exist).

(5) Translate the function $q_0(x) = 2x^3 + x$ by $\binom{3}{20}$ and confirm the x-coordinate of the PoI of the translated function.

Solution

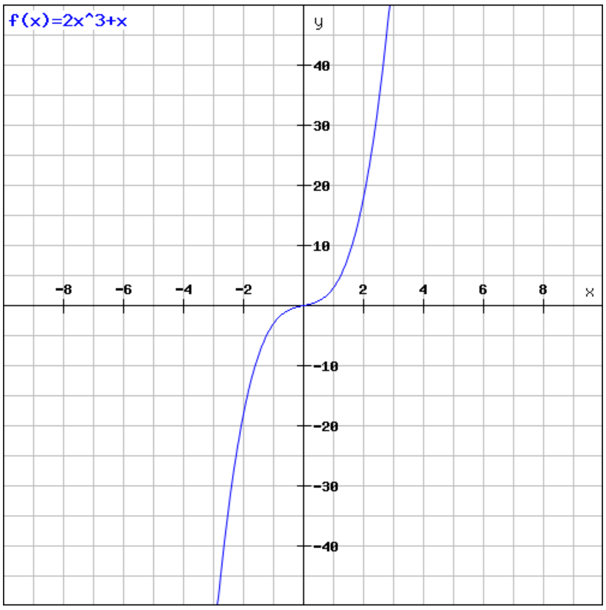
Translated function is $q(x) = 2(x-3)^3 + (x-3) + 20$

ie $q(x) = 2(x^3 - 9x^2 + 27x - 27) + x + 17$

$= 2x^3 - 18x^2 + 55x - 37$

From (2), the PoI is at $x = -\frac{b}{3a} = -\frac{(-18)}{3(2)} = 3$

(as expected, as the function has been translated by 3 to the right).



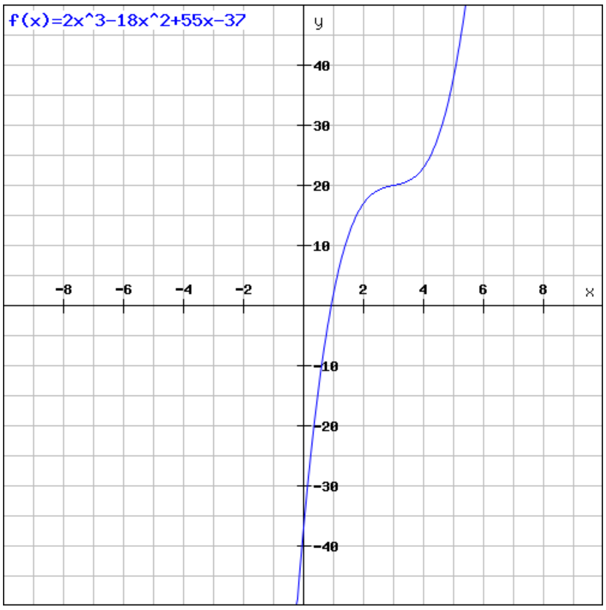


Fig. 4: $q_0(x) = 2x^3 + x$ Fig 5: $q(x) = 2x^3 - 18x^2 + 55x - 37$

(6) Find the function $p_0(x)$ that results from translating

$p(x) = x^3 - 6x^2 - 4x + 40$, so that its PoI is at the Origin. Sketch $p_0(x)$

Solution

The PoI of $p(x)$ is at $(2, 16)$

Translating $p(x)$ by $\binom{-2}{-16}$ gives

$p_0(x) = (x+2)^3 - 6(x+2)^2 - 4(x+2) + 40 - 16$

$= (x^3 + 6x^2 + 12x + 8) - 6(x^2 + 4x + 4) - 4x + 16$

$= x^3 - 16x = x(x^2 - 16) = x(x-4)(x+4)$

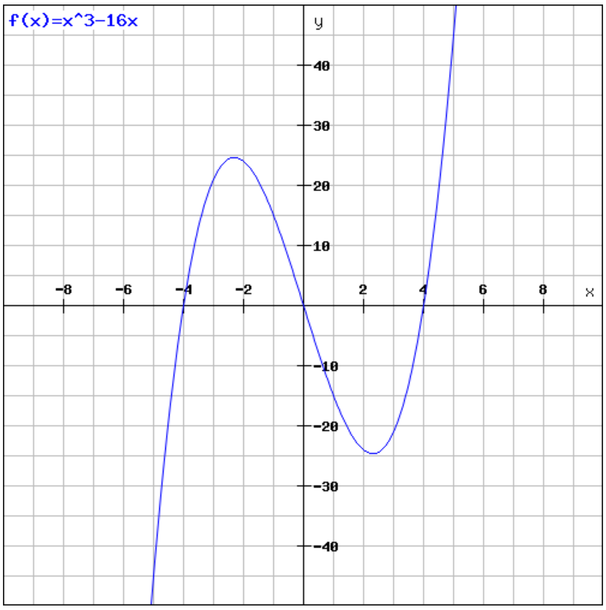


Fig. 6: $p_0(x) = x^3 - 16x$

(7) Consider $g(x) = x^3 + bx^2 + cx + d$

Translate the graph of $g(x)$ so that its PoI is at the Origin (to give $g_0(x)$).

Solution

The PoI of $g(x)$ is at $(-\frac{b}{3}, -\frac{b^3}{27} + \frac{b^3}{9} - \frac{bc}{3} + d)$

ie $(-\frac{b}{3}, \frac{2b^3}{27} - \frac{bc}{3} + d)$

Translating $g(x)$ by $\begin{pmatrix} \frac{b}{3} \\ -\frac{2b^3}{27} + \frac{bc}{3} - d \end{pmatrix}$ gives

$g_0(x) = (x - \frac{b}{3})^3 + b(x - \frac{b}{3})^2 + c(x - \frac{b}{3}) + d - \frac{2b^3}{27} + \frac{bc}{3} - d$

$= x^3 + 3x^2\left(-\frac{b}{3}\right) + 3x\left(-\frac{b}{3}\right)^2 + \left(-\frac{b}{3}\right)^3$

$+b\left(x^2 - \frac{2xb}{3} + \frac{b^2}{9}\right) + cx - \frac{bc}{3} - \frac{2b^3}{27} + \frac{bc}{3}$

$= x^3 + x\left(\frac{b^2}{3} - \frac{2b^2}{3} + c\right) - \frac{b^3}{27} + \frac{b^3}{9} - \frac{2b^3}{27}$

$= x^3 + (c - \frac{b^2}{3})x$

Check:

For $p(x) = x^3 - 6x^2 - 4x + 40$ [in (6)], we found that

$p_0(x) = x^3 - 16x$

For $p(x), c - \frac{b^2}{3} = -4 - \frac{(-6)^2}{3} = -4 - 12 = -16$

(8) Consider the effect of reflecting $g_0(x) = x^3 + c_0x$ in the x-axis and then in the y-axis. What does this reveal?

Solution

A reflection in the x-axis gives $-(x^3 + c_0x)$.

A subsequent reflection in the y-axis then gives

$-\left((-x)^3 + c_0(-x)\right) = x^3 + c_0x$; ie the original function.

The effect of the two reflections is a rotation of 180° about the Origin, so $g_0(x) = x^3 + c_0x$ has rotational symmetry (of order 2) about the PoI at the Origin.

As the PoI can always be translated to the Origin, this means that all cubics have rotational symmetry about their PoI. This confirms that the PoI lies midway between any turning points.

(9) Show that the PoI of $y = a(x-p)(x-q)(x-r)$ is at

$x = \frac{1}{3}(p+q+r)$

Solution

At the PoI, $x = -\frac{b}{3a}$

Equating coefficients of x^2 in

$ax^3 + bx^2 + cx + d = a(x-p)(x-q)(x-r)$,

$b = a(-p-q-r)$, so that $-\frac{b}{3a} = \frac{1}{3}(p+q+r)$

(10) By considering the gradient at the Origin, sketch the possible shapes of $g_0(x) = x^3 + (c - \frac{b^2}{3})x$

Solution

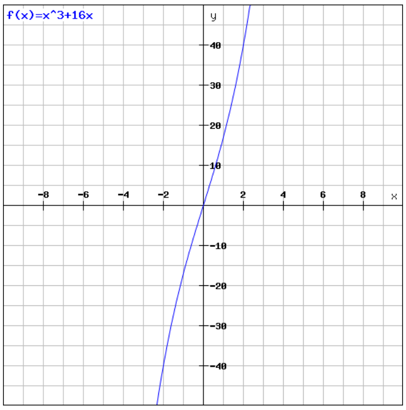


Fig. 7: $y = x^3 + 16x$

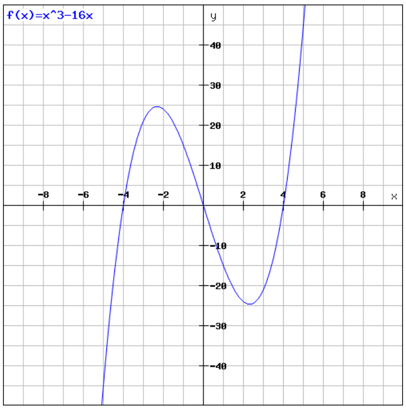


Fig. 8: $y = x^3 - 16x$

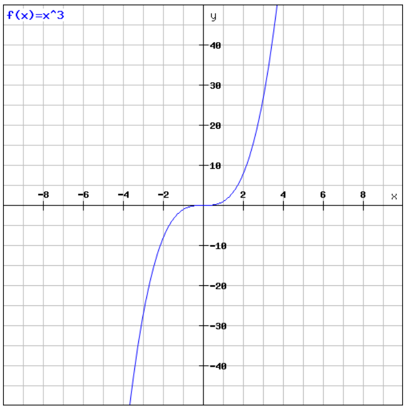


Fig. 9: $y = x^3$

As a cubic can be translated so that its PoI is at the Origin, the above shapes also apply to any function of the form

$g(x) = x^3 + bx^2 + cx + d$

(11) Consider $f(x) = ax^3 + bx^2 + cx + d$

Find $f_0(x)$, the translated function with its PoI at the Origin.

Solution

The PoI is at $(-\frac{b}{3a}, D)$, where $D = f\left(-\frac{b}{3a}\right)$

and $f_0(x) = a(x - \frac{b}{3a})^3 + b(x - \frac{b}{3a})^2 + c(x - \frac{b}{3a}) + d - D$

Because $f_0(0) = f_0''(0) = 0$, $f_0(x)$ must take the form

$ax^3 + c_1 x$

where $c_1 = a(3)\left(-\frac{b}{3a}\right)^2 + b(2)\left(-\frac{b}{3a}\right) + c = \frac{b^2}{3a} - \frac{2b^2}{3a} + c$

$= c - \frac{b^2}{3a}$

So $f_0(x) = ax^3 + \left(c - \frac{b^2}{3a}\right)x$,

which agrees with (7) when $a = 1$

Check:

From (5), we know that $q(x) = 2x^3 - 18x^2 + 55x - 37$
translates to $q_0(x) = 2x^3 + x$

Here $c - \frac{b^2}{3a} = 55 - \frac{(-18)^2}{3(2)} = 55 - 54 = 1$, which agrees with
$q_0(x) = 2x^3 + x$

(12) What condition must apply to b & c for the function

$f(x) = ax^3 + bx^2 + cx + d$ to have two turning points?

Solution

Method 1

$f'(x) = 3ax^2 + 2bx + c$

We require $f'(x) = 0$ to have two (distinct) solutions, so $(2b)^2 - 4(3a)c > 0$

ie $b^2 > 3ac$

Method 2

We require $f_0'(0) < 0$ (see Fig. 8 earlier)

so that $c - \frac{b^2}{3a} < 0$, and hence $b^2 > 3ac$ again.

(13) Another option for sketching cubics

Consider the function $y = x^3 + 5x^2 + 6x + 2$

This can be written as

$y = x(x^2 + 5x + 6) + 2 = x(x + 2)(x + 3) + 2$

and can be sketched by translating $y = x(x + 2)(x + 3)$

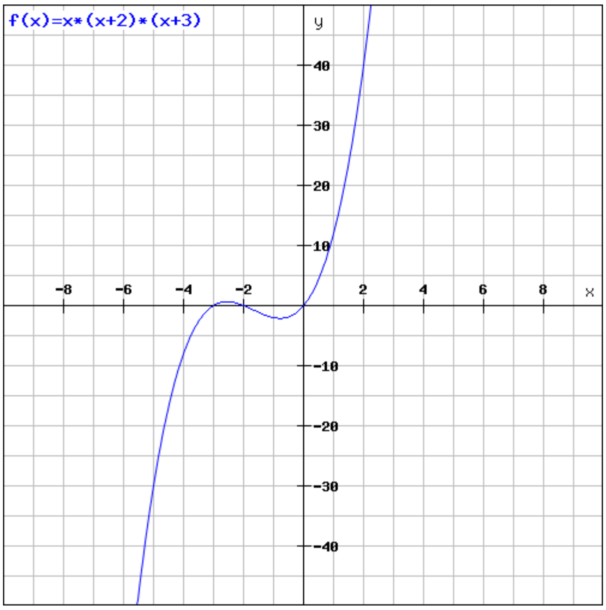


Fig. 10: $y = x(x + 2)(x + 3)$

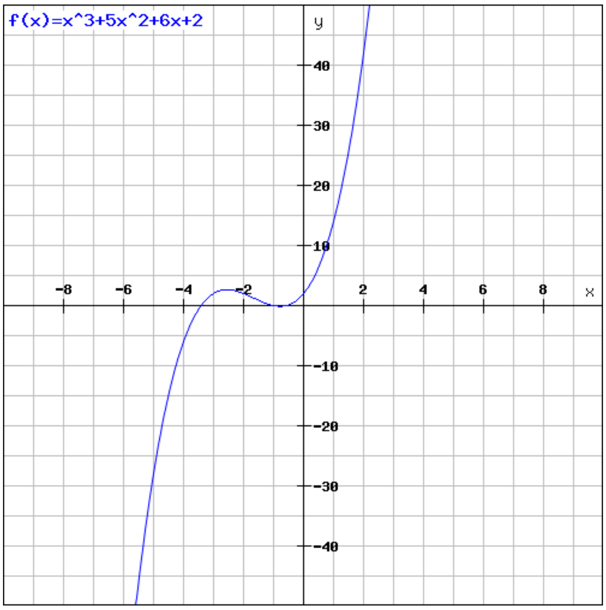


Fig. 11: $y = x^3 + 5x^2 + 6x + 2$

(14) Summary

For $f(x) = ax^3 + bx^2 + cx + d$

(1) There is always one point of inflexion, at $x = -\frac{b}{3a}$

(2) Cubic curves have rotational symmetry (of order 2) about the PoI.

(3) The (x-coordinate of the) PoI lies midway between any turning points.

(4) The (x-coordinate of the) PoI is the average of the roots, when there are 3 real roots (and also when there are complex roots).

(5) The shape of the curve $f(x) = ax^3 + bx^2 + cx + d$ can be established by translating its PoI to the Origin, to give

$f_0(x) = ax^3 + \left(c - \frac{b^2}{3a}\right)x$

(6) There will be two turning points when $f_0'(0) < 0$

(see Fig. 8 above); ie when $b^2 > 3ac$