**Cubics** (5 pages; 5/6/23)

[See also "Turning Points & Points of Inflexion".]

($f(x) = ax^3 + bx^2 + cx + d$ throughout)

(1) Cubics always have (exactly) one point of inflexion:

$f'(x) = 3ax^2 + 2bx + c$ and $f''(x) = 6ax + 2b$

So $f''(x) = 0 \Rightarrow x = -\frac{b}{3a}$

[For a general function, $f''(x) = 0$ is a necessary (but not sufficient) condition for a point of inflexion (which is a turning point of the gradient). However, for a cubic it is a sufficient condition as well.]

(2) Without loss of generality, we can consider a cubic that has its point of inflexion at the Origin, and is therefore of the form

$y = g(x) = ax^3 + cx$.

As $g(-x) = -g(x)$ [ie $g(x)$ is an odd function], there is rotational symmetry (of order 2) about the point of inflexion.

The symmetry implies that the point of inflexion is halfway between the turning points (if they exist).

(3) Cubics can be classified by the number of stationary points.

As $f'(x) = 3ax^2 + 2bx + c$, there will be 0, 1 or 2 stationary points, according to whether $(2b)^2 - 4(3a)c$ is negative, zero or positive; ie whether if $b^2 - 3ac$ is negative, zero or positive.

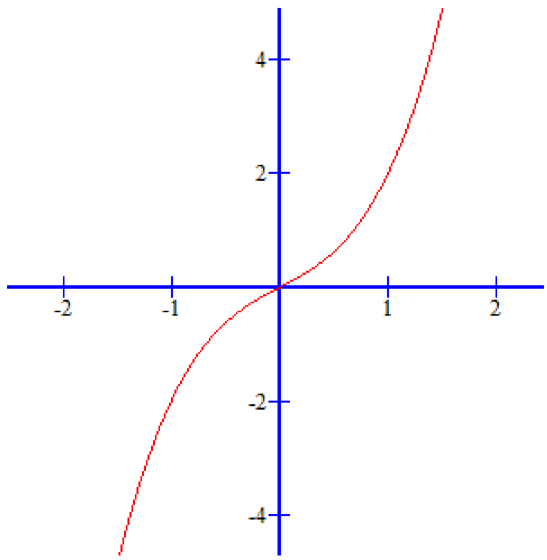
So there are 3 types:

(i) When $b^2 - 3ac < 0$ (no stationary points)

eg $y = x^3 + x = x(x^2 + 1)$: only real root is $x = 0$

$\frac{dy}{dx} = 3x^2 + 1 > 0$ (so no stationary points)

$\frac{d^2y}{dx^2} = 6x$ : point of inflexion at $x = 0$

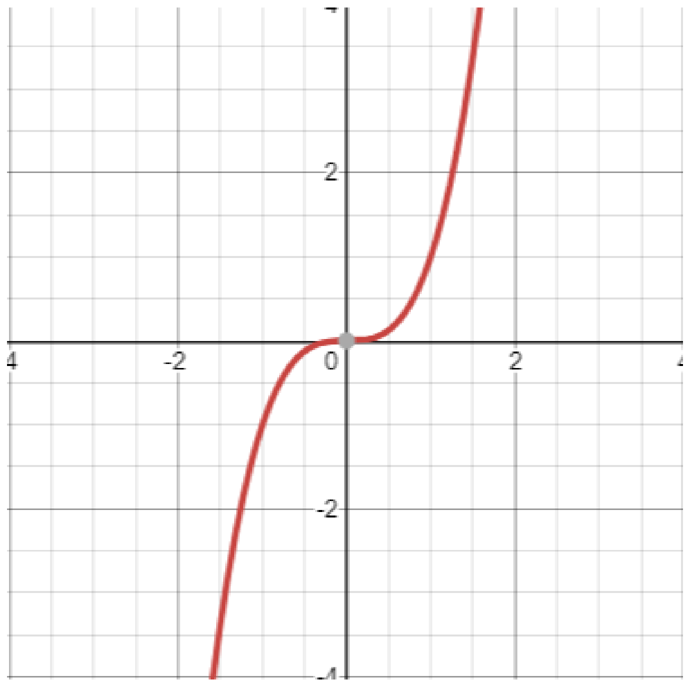



(ii) When $b^2 - 3ac = 0$ (1 stationary point; no turning points)

eg $y = x^3$

$\frac{dy}{dx} = 3x^2$ : stationary point at $x = 0$, and $\frac{dy}{dx} > 0$ elsewhere

$\frac{d^2y}{dx^2} = 6x$: point of inflexion at $x = 0$



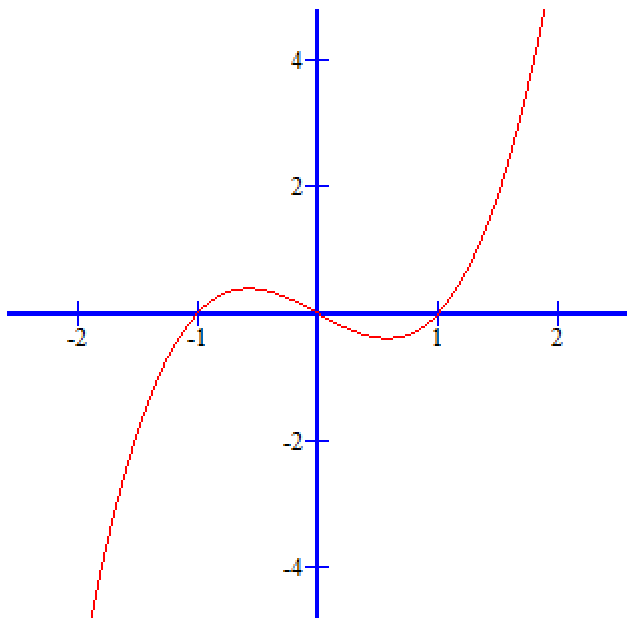

(iii) When $b^2 - 3ac > 0$ (2 stationary (turning) points)

eg $y = x^3 - x = (x+1)x(x-1)$

$\frac{dy}{dx} = 3x^2 - 1$: turning points at $x = \pm\frac{1}{\sqrt{3}}$

and $\frac{dy}{dx} < 0$ at $x = 0$ (this in itself means that there must be two turning points)

$\frac{d^2y}{dx^2} = 6x$ : point of inflexion at $x = 0$




(4) The point of inflexion of $y = a(x-p)(x-q)(x-r)$ is at $x = \frac{1}{3}(p+q+r)$

**Proof**

At the PoI, $x = -\frac{b}{3a}$

Equating coefficients of $x^2$ in

$ax^3 + bx^2 + cx + d = a(x-p)(x-q)(x-r)$ ,

$b = a(-p-q-r)$, so that $-\frac{b}{3a} = \frac{1}{3}(p+q+r)$

(5) Transformations

(i) The cubic $y = x^3 + 2x^2 + x + 3$ can be sketched by translating $y = x(x^2+2x+1) = x(x+1)^2$ by $\binom{0}{3}$

(ii) The cubic $y = x^3 + 3x^2 + x + 1$ has its point of inflexion at $(-1, 2)$. In order to translate it by $\binom{1}{-2}$, so that the point of inflexion is at the Origin, we write

$y + 2 = (x-1)^3 + 3(x-1)^2 + (x-1) + 1$

so that $y = x^3 + x^2(-3+3) + x(3-6+1) - 1 + 3 - 2$

and $y = x^3 - 2x$